Money, Fiscal Defecits and Government Debt in a Monetary Union

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Publication date:
1996

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Abstract

The replacement of national currencies by a common currency in the EMU causes a monetary externality if the European Central Bank is inclined to monetize part of outstanding government debt in the community. High government debt in one part of the EU then increases the common inflation rate. We model debt stabilization in the EU as a differential game between fiscal authorities and the ECB. Three different equilibria are considered: the Nash open-loop equilibrium, the Stackelberg open-loop equilibrium with the ECB leading and the Stackelberg open-loop equilibrium with the fiscal authorities leading. Dynamics of the fiscal deficits, inflation and government debt in a monetary union are derived and compared with an EU with national monetary policies.

January 1996

JEL code: C73, E58, E52, E63, F33, F42
Keywords : European Monetary Union, European Central Bank, Monetary Policy, Fiscal Policy, Differential Game Theory

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Introduction

The completing of the European Monetary Union (EMU) by 1999 with the establishment of an independent common central bank, the European Central Bank (ECB), that manages the common monetary policy has considerable economic, institutional and political consequences[3]. Concerning the economic consequences, the ECB will control a stock of European base money of some 600 bln $ and will be a major player at the European Union (EU) level and even at a global level. Its monetary policy will have a major influence on inflation and the business cycle in the EU. Political aspects of the ECB concern the question how individual countries and interests will influence the decision-making process in the ECB and in particularly how independent the ECB will be. In this paper we analyse the public finance consequences of ECB monetary policies: its decisions on monetization and lending to fiscal authorities of individual countries has potentially a considerable impact on public finance of the participating countries.

The Delors Report (1989) expressed a fear of excessive government debt accumulation in the European Monetary Union: without binding rules on fiscal policies, unsound public finance in some EU countries will endanger the credibility of low-inflation policies of the ECB. A forced monetization of fiscal deficits by the ECB to ’bail-out’ undisciplined national fiscal authorities transfers the adjustment burden of unsound public finance in a part of the EU to the entire EU in the form of a higher European inflation rate. The design of fiscal policy will largely remain a national issue in the EMU due to the ’subsidiarity principle’: the design and implementation of fiscal policy should as much as possible be delegated to the different countries. An additional source of pressure on the ECB to relax monetary policies could come from the European Commission that might consider borrowing from the ECB a less troublesome way to generate revenues than increasing contributions of the member states. EC (1993) addresses the current problems in the EU with fiscal discipline in detail.

Alesina and Grilli (1993) and von Hagen and Sűppel (1994) analyse the conduct of ECB monetary policy from a stabilization perspective. Garrett (1993) discusses the

[3] See e.g. Sarcinelli (1992) or Sardelis (1993) for more details on the institutional design and operating of the ECB.

political aspects of the EMU and ECB policies. Levine (1993) and Levine and Brociner (1994) analyse the effects of fiscal policy coordination in the EMU. To avoid a deficit and inflationary bias in the EMU, the Delors Committee proposes limits on fiscal deficits and government debt and a high degree of ECB independence. This paper stresses the public finance consequences of monetary policies pursued by the ECB.

A conflict between national fiscal authorities and the ECB arises on whether fiscal or monetary instruments should be adjusted to stabilize government debt in the EMU. Tabellini (1986) formulated a differential game on government debt stabilization between a fiscal and a monetary authority in a closed economy setting. This paper applies the Tabellini (1986) framework to a monetary union with a common central Bank building on the results in van Aarle, Bovenberg and Raith (1995a) and (1995b).

The analysis is organized as follows: section 2 describes a differential game on debt stabilization between a fiscal authority and a national monetary authority. Three different non-cooperative solution concepts are explored: the Nash equilibrium in which all players act simultaneously, the Stackelberg equilibrium with the monetary authority acting as a Stackelberg leader and the Stackelberg equilibrium with the fiscal authority acting as Stackelberg leader. Section 3 considers the problem of government debt stabilization in a monetary union in which national monetary authorities have been replaced by a single monetary authority. Section 4 compares the outcomes under national monetary policy -section 2- with a monetary union -section 3. Section 5 considers a numerical example to illustrate the results from the theoretical part.

2. Debt stabilization with national monetary policy

The countries that enter the EMU do so from different starting positions. Table 2.1 provides data on gross government indebtedness, $d(t)$, interest payments on government debt, $rd(t)$, primary fiscal deficits, $f(t)$, seigniorage $m(t)$ -all expressed of fractions of GDP- and inflation rates, $\pi(t)$, in the different EU countries. The last column gives the share, ω, of the different countries in aggregate EU GDP.

Table 1 Key-indicators public finance in the EU

	$d(t)$	$rd(t)$	$f(t)$		$m(t)$		π		ω
	94	94	79-89	90-94	79-89	90-94	78-89	90-94	94
BG	141.6%	9.4%	0.9%	-3.2%	0.2%	0.0%	5.0%	3.1%	3,1%
DK	69.7%	3.1%	-0.1%	0.0%	0.4%	0.6%	7.6%	2.6%	2,0%
FR	62.1%	3.3%	0.3%	1.0%	0.5%	-0.1%	8.2%	2.9%	17,5%
GE	50.8%	3.4%	0.1%	0.4%	0.5%	0.6%	3.0%	3.4%	22,9%
GR	103.7%	15.7%	5.3%	2.2%	3.5%	2.8%	19.4%	16.8%	1,1%
IR	89.6%	5.8%	3.4%	-3.5%	0.8%	0.3%	10.1%	3.0%	0,7%
IT	116.5%	10.2%	4.1%	-0.3%	1.9%	0.9%	12.2%	6.3%	16,8%
NL	81.9%	4.8%	0.6%	-0.7%	0.6%	0.5%	3.3%	2.5%	4,3%
PO	70.2%	6.3%	-0.7%	-1.7%	4.5%	3.0%	19.2%	11.6%	1,1%
SP	62.6%	5.3%	2.4%	1.4%	3.0%	1.0%	12.0%	5.8%	7,7%
UK	52.3%	3.0%	-0.9%	2.7%	0.3%	0.2%	8.0%	6.7%	14,7%
AU	58.6%	3.5%	0.4%	-0.7%	0.5%	0.5%	3.9%	3.4%	2,5%
FI	60.0%	4.3%	-0.9%	1.6%	1.0%	0.7%	7.5%	4.3%	1,9%
SW	80.5%	2.8%	0.5%	5.5%	0.7%	1.5%	8.2%	6.6%	3,5%
EU	72.1%	5.0%	1.0%	0.8%	1.0%	0.5%	7.7%	4.5%	100

$d(t)$, $rd(t)$ and ω are measured at the end of 1994. $m(t)$, $f(t)$ and π are 1979-1989 and 1990-1994 averages. Source: OECD (1994). A negative primary fiscal deficit implies a primary fiscal surplus. Seigniorage is calculated as the change in base money, $M(0)$ divided by GDP. BG=Belgium, DK=Denmark, FR=France, GE=Germany, GR=Greece, IR=Ireland, IT=Italy, NL=the Netherlands, PO=Portugal, SP=Spain, UK=United Kingdom, AU=Austria, FI=Finland, SW=Sweden.

Considerable differences in (gross) government indebtedness (column 2) and the burden from interest payments (column 3) exist currently. Primary fiscal deficits (column 4 and 5), seigniorage (column 6 and 7) and inflation (column 8) also display considerable variety both across countries and over time. The EU average found in the last row can be used to divide the EU into two parts: a part that is more and a part that is less heavily indebted than the EU average. The first part encompasses Belgium, Greece, Ireland, Italy, the Netherlands and Sweden while the other countries have below average government debt. When looking at inflation, Belgium and the Netherlands move to the below EU average group, whereas Portugal, Spain and the UK move to the above average group. Roughly speaking, a division between the Northern and the Southern part of the EU is present. On average, the Northern part is characterized by lower fiscal deficits, government indebtedness and inflation than the Southern part. Doubts are often raised whether the EU will satisfy the fiscal convergence criteria in 1999, when the introduction of the common currency is planned[4]. Corsetti and Roubini (1993) test the sustainability of the process of government debt accumulation in the EU. They find that the current process of government debt accumulation in Italy, Belgium and Ireland is not sustainable in the long run.

Tabellini (1986) develops a differential game between a fiscal and a monetary authority on stabilizing government debt in a national setting. van Aarle, Bovenberg and Raith (1995b) extend the Tabellini (1986) analysis with national monetary policy. This paper uses the setting with national monetary policies to compare debt stabilization in a monetary union with debt stabilization in case of national monetary policies, representing the pre-EMU situation where every country implements a national monetary policy.

In a national setting, the dynamic government budget constraint relates the change in the government debt to GDP ratio, $\dot{d}$ (a dot above a variable denotes its time derivative), to interest payments, $rd(t)$, the primary fiscal deficit, $f(t)$, of the fiscal authority and money creation or seigniorage, $m(t)$, of the monetary authority:

$$\dot{d} = rd(t) + f(t) - m(t) \tag{1}$$

in which r represents the interest rate minus the growth rate of output. We assume that r is given and independent of the amount of government debt at time t. Alesina e.a. (1994) investigate default risk premia on government debt in the OECD and find that risk premia have been absent or only small. Money creation and inflation are positively related as long as the economy is on the increasing part of the seigniorage Laffer curve[5].

If the primary deficit plus the interest payments exceeds the seigniorage revenues generated by the monetary authority, government debt accumulation allows policymakers to shift the adjustment burden to the future. Fiscal consolidation, therefore, can be achieved in two manners: increasing monetization or reducing primary fiscal deficits. As both instruments are delegated to different authorities, a conflict arises between the

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[4] A steady-state debt ratio of 60% of GDP results from primary fiscal deficits of 3% of GDP and a net of output growth real interest rate of 5%, as the Delors Committee assumed when advocating such debt and deficit targets. As the simulations will show, however, it can take a fairly long time before new steady-states are achieved. The quick convergence implicitly assumed by the Delors Committeer seems to be rather optimistic.

[5] At the increasing part of the Laffer curve the interest elasticity of money demand is less than 1 in absolute value. Empirical studies on money demand in industrial countries (see e.g. Boughton (1991)) indicate that inflation in industrial countries is indeed well below the seigniorage maximizing rate of inflation.

monetary authority and the fiscal authority about the division of the adjustment burden from fiscal consolidation. Government solvency is ensured if the following transversality condition -generally referred to as the no-Ponzi game condition[6]- is met:

$$\lim_{t \to \infty} d_i(t) e^{-rt} = 0 \qquad i = 1,2 \tag{2}$$

Following Tabellini (1986), we formalize the strategic interaction between monetary and fiscal authorities by specifying instruments, objectives and the game structure. Consider the following intertemporal loss function of the fiscal authority, which depends on the time profiles of the primary fiscal deficit and government debt:

$$L^F(t_0) = \frac{1}{2} \int_{t_0}^{\infty} \{(f(t) - \bar{f})^2 + \lambda (d(t) - \bar{d}^F)^2\} e^{-\delta (t - t_0)} dt \tag{3}$$

The control variable of the fiscal authority of country 1 is the primary fiscal deficit $f(t)$[7]. The minimization problem of the fiscal authorities is carried out subject to the dynamic government budget constraint, (1), the transversality condition on government debt (2) and the initial stock of government debt, $d(0)$. $\bar{f}$ and $\bar{d}^F$ represent the primary fiscal deficit and government debt targets of the fiscal authority. If $\bar{f}$ and $\bar{d}^F$ are positive, fiscal policies will tend to exhibit a fiscal deficit bias[8]. These fiscal targets reflect the institutional and political structures in which decision making on fiscal policies takes place and are assumed to be given in the remainder of the analysis. λ can be looked upon as the degree of fiscal discipline of the fiscal authority as it gives the weight that the fiscal authorities attach to government debt stabilization.

The subjective rate of time preference, δ, determines how much future losses are discounted by policymakers. If $\delta > r$, the subjective costs of an additional unit of debt are lower than their objective costs and additional debt is preferred by policymakers. In the

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6 Empirical studies on government solvency -using the implications of the No-Ponzi game condition- in Europe are found in Grilli (1988) and Corsetti and Roubini (1993). Baglioni and Cherubini (1993) study the case of Italy.

7 von Hagen (1992) analyses the budgeting procedures in the EU.

8 See von Hagen and Harden (1995) on the emergence of “fiscal illusion” in the political system, inducing a bias towards fiscal deficits and accumulation of government debt.

remainder of the analysis we assume that such a form of impatientness is present. As in Tabellini (1986), government debt features in the loss functions because higher levels of debt imply larger tax distortions to service interest payments. Moreover, the larger the stock of public debt, the larger the required adjustments in taxes associated with fluctuations in the interest rate and the growth rate of real output. A high level of public debt is also likely to crowd out private investment, and it may induce undesirable intergenerational redistributions of wealth, if Ricardian equivalence does not hold.

Monetary policy by the monetary authority is implemented such as to minimize the following intertemporal loss function, $L^M(t_0)$:

$$L^M(t_0) = \frac{1}{2}\int_{t_0}^{\infty} \{(m(t) - \bar{m})^2 + \tau(d(t) - \bar{d}^M)^2\} e^{-\delta(t-t_0)} dt \tag{4}$$

We concentrate on three different non-cooperative equilibria of the differential game between the monetary and fiscal authority on debt stabilization. The three equilibria are the Nash open-loop equilibrium, in which both players act simultaneously, the Stackelberg open-loop equilibrium with the monetary authority acting as Stackelberg leader and the Stackelberg equilibrium with the fiscal authority acting as Stackelberg leader. We concentrate on open-loop strategies instead of subgame-perfect closed-loop strategies as the latter do not allow to derive an analytical solution and one has to rely on numerical simulation[9]. Details on these equilibria in non-cooperative differential games are found in Basar and Olsder (1982)[10].

The Nash open-loop equilibrium is found by solving the dynamic optimization problems of both players simultaneously. The present value Hamiltonian of the fiscal authority, $H^F(t)$, is given by,

[9] For details on the Nash closed-loop equilibria of the debt stabilization game, see Tabellini (1986). van Aarle, Bovenberg and Raith (1995a) compare the open-loop and closed-loop Nash equilibria with national monetary policy and a monetary union. The cooperative equilibrium is extensively studied in van Aarle, Bovenberg and Raith (1995b).

[10] The Nash equilibria are time-consistent but the open-loop equilibrium is not subgame perfect like the closed-loop equilibrium. The Stackelberg equilibria are not time-consistent and require therefore the presence of a commitment technology, e.g. reputational forces. We assume indeed that such commitment technologies are available in the dynamic debt stabilization game between monetary and fiscal authorities.

$$H^F(t) = \frac{1}{2}(f(t) - \bar{f})^2 + \frac{\lambda}{2}(d(t) - \bar{d}^F)^2 + \mu^F(t)(rd(t) + f(t) - m(t)) \tag{5}$$

is which $\mu^F(t)$ denotes the co-state variable attached in the optimization problem to government debt of country 1 at time t. It is sometimes referred to as the "cost of public funds" as it measures the costs from an additional unit of government debt that requires higher future taxes to pay its amortization. This co-state variable is an important variable: the concern about government debt stabilization that it reflects, determines the actual policies that players pursue at each point in time. The first order conditions of this dynamic optimization problem characterize the optimal policies of the different players. The first order conditions from the optimization problem of the fiscal authority are given by:

$$\begin{gathered} f(t) = \bar{f} - \mu^F(t) \\ \dot{\mu}^F = -\lambda(d(t) - \bar{d}^F) + (\delta - r)\mu^F(t) \end{gathered} \tag{6}$$

The present value Hamiltonian of the monetary authority $H^M(t)$,

$$H^M(t) = \frac{1}{2}(m(t) - \bar{m})^2 + \frac{\tau}{2}(d(t) - \bar{d}^M)^2 + \mu^M(t)(rd(t) + f(t) - m(t)) \tag{7}$$

is minimized if,

$$\begin{gathered} m(t) = \bar{m} + \mu^M(t) \\ \dot{\mu}^M = -\tau(d(t) - \bar{d}^M) + (\delta - r)\mu^M(t) \end{gathered} \tag{8}$$

(6) and (8) show how the desire to stabilize government debt, as measured by the co-state variables $\mu^i(t)$ influences fiscal and monetary policies. Together, (6) and (8) determine the Nash open-loop equilibrium and can be combined to a system of linear differential equations describing the dynamics of government debt, $d(t)$, and the co-state variable(s) associated with government debt, $\mu^i(t)$. This system of linear differential equations is analysed in Appendix A. The second column of table 2 provides steady state government debt, primary fiscal deficit and money creation of the Nash open-loop equilibrium.

In the Nash equilibrium all players implement their policies simultaneously. In the Stackelberg equilibrium one player, the Stackelberg leader, obtains a more dominant role, enabling it to set its policies before the other player(s). By moving first and considering the preferred policy of the other player, the Stackelberg leader has a strategic advantage that enables it to shift most of the adjustment burden from debt stabilization to the other player and to have a better performance with respect to the other objective(s). A strong and independent central bank could have such a strategic advantage in the dynamic interaction with the fiscal authority on the issue of government debt stabilization. On the other hand, the fiscal player can be Stackelberg leader in the government debt stabilization game with a weak and dependent central bank[11].

In case of Stackelberg leadership of the monetary authority, the monetary player considers in its decision problem that the fiscal authorities react according to (6). If it can act as a Stackelberg leader towards the fiscal player, its present value Hamiltonian becomes:

$$\begin{aligned} H^M(t) = \frac{1}{2}(m(t)-\overline{m})^2 + \frac{\tau}{2}(d(t)-\overline{d}^M)^2 + \mu^M(t)(rd(t)+\overline{f}-\mu^F(t)-m(t)) \\ + \rho^M(t)(-\lambda(d(t)-\overline{d}^M)) + (\delta - r)\mu^F(t)) \end{aligned} \tag{9}$$

in which $\rho^M(t)$ is a co-state variable, attached to the co-state variable, $\mu^F(t)$, of the Stackelberg follower, the fiscal authority. The first-order conditions that result from minimizing (9) are:

$$\begin{aligned} m(t) &= \overline{m} + \mu^M(t) \\ \dot{\mu}^M &= -\tau(d(t)-\overline{d}^M) + (\delta - r)\mu^M(t) + \lambda\rho^M(t) \\ \dot{\rho}^M &= r\rho^M(t) + \mu^M(t) \end{aligned} \tag{10}$$

(6) and (10) together describe the policies in the Stackelberg equilibrium with Stackelberg leadership of the monetary authority. (10) reveals that the monetary authority considers the desire of the fiscal authority to stabilize government debt, as summarized by λ, when deciding on optimal monetary policy. Appendix A provides the dynamic system $\{d(t),\mu^F(t),\mu^M(t),\rho^M(t)\}$ that characterizes the Stackelberg open-loop equilibrium with Stackelberg leadership of the monetary authority. The third column of table 2 describes the steady-state of the Stackelberg open-loop equilibrium with leadership of the monetary player, i.e. an independent central bank.

Consider next the case where the fiscal authority obtains Stackelberg leadership in the debt stabilization game with the monetary authority: a situation with a dependent central bank. When selecting its preferred fiscal policy, the fiscal authority considers that the monetary authority sets its policy according to (8). The first-order conditions governing optimal fiscal policy with Stackelberg leadership change from (6) into:

[11] See Grilli, Masciandaro and Tabellini (1991) for empirical evidence on central bank independence in OECD countries.

$$
\begin{aligned}
f(t) &= \bar{f} - \mu^F(t) \\
\dot{\mu}^F &= -\lambda(d(t) - \bar{d}^F) + (\delta - r)\mu^F(t) + \tau\rho^F(t) \\
\dot{\rho}^F &= r\rho^F(t) + \mu^F(t)
\end{aligned}
\tag{11}
$$

$\rho^F(t)$ is an additional co-state variables attached by the fiscal player to the co-state variables of the monetary player, $\mu^M(t)$. (11) shows that the fiscal authority when acting as Stackelberg leader considers the desire -reflected by the preference parameter τ- of the monetary authority to stabilize government debt by increasing money creation. With (8), (11) describes the Stackelberg open-loop equilibrium with the fiscal authority acting as Stackelberg leader towards the monetary authority. The steady-state of the Stackelberg open-loop equilibrium with Stackelberg leadership of the fiscal authorities, i.e. a dependent central bank, is found in the final column of table 2.

Table 2 Steady-state with national monetary policy

	Nash open-loop	Stackelberg open-loop with monetary leadership	Stackelberg open-loop with fiscal leadership
$d(\infty)$	$\frac{(\delta-r)^2(\bar{f}-\bar{m})+(\delta-r)\lambda\bar{d}^F+(\delta-r)\tau\bar{d}^M}{\Delta_N}$	$\frac{(\delta-r)(r(\delta-r)-\lambda)(\bar{f}-\bar{m})+(r(\delta-r)-\lambda)\lambda\bar{d}^F+r(\delta-r)\tau\bar{d}^M}{\Delta_M}$	$\frac{(\delta-r)(r(\delta-r)-\tau)(\bar{f}-\bar{m})+r(\delta-r)\lambda\bar{d}^F+(r(\delta-r)-\tau)\tau\bar{d}^M}{\Delta_F}$
$f(\infty)$	$\bar{f}-\frac{\lambda[(\delta-r)(\bar{f}-\bar{m})+(r(\delta-r)-\tau)\bar{d}^F+\tau\bar{d}^M]}{\Delta_N}$	$\bar{f}-\frac{\lambda[(r(\delta-r)-\lambda)(\bar{f}-\bar{m})+r(r(\delta-r)-\lambda-\tau)\bar{d}^F+\tau r\bar{d}^M]}{\Delta_M}$	$\bar{f}-\frac{\lambda r[(\delta-r)(\bar{f}-\bar{m})+(r(\delta-r)-\tau)\bar{d}^F+\tau\bar{d}^M]}{\Delta_F}$
$m(\infty)$	$\bar{m}+\frac{\tau[(\delta-r)(\bar{f}-\bar{m})+\lambda\bar{d}^F+(r(\delta-r)-\lambda)\bar{d}^M}{\Delta_N}$	$\bar{m}+\frac{\tau r[(\delta-r)(\bar{f}-\bar{m})+\lambda\bar{d}^F+(r(\delta-r)-\lambda)\bar{d}^M]}{\Delta_M}$	$\bar{m}+\frac{\tau[(r(\delta-r)-\tau)(\bar{f}-\bar{m})+r\lambda\bar{d}^F+r(r(\delta-r)-\lambda-\tau)\bar{d}^M]}{\Delta_F}$
Δ	$\Delta_N=[\lambda+\tau-r(\delta-r)](\delta-r)$	$\Delta_M=[2\lambda+\tau-r(\delta-r)]r(\delta-r)-\lambda^2$ $=r\Delta_N-\lambda(\lambda-r(\delta-r))$	$\Delta_F=[\lambda+2\tau-r(\delta-r)]r(\delta-r)-\tau^2$ $=r\Delta_N-\tau(\tau-r(\delta-r))$

To avoid dynamic instability and violation of the no-Ponzi game condition, we impose the condition that the dynamic systems that describe Nash and Stackelberg equilibria are saddlepoint stable[12]. Stability is ensured if Δ is positive in case of the Nash equilibrium and negative in the Stackelberg equilibria. The stability conditions (see table 2) for the Stackelberg equilibria are more strict than in the Nash equilibrium where $\lambda+\tau>r(\delta-r)$ suffices, given our assumption that $\delta>r$. Stability in the Stackelberg equilibrium with monetary leadership requires that $\lambda>r(\delta-r)$ whereas stability in the Stackelberg equilibrium with fiscal leadership requires that $\tau>r(\delta-r)$. These conditions imply that stability of the Stackelberg equilibria holds only if the Stackelberg follower cares sufficiently about government debt stabilization.

A comparison of the three non-cooperative equilibria leads to the following proposition:

Proposition 1

(a) Steady-state government debt is lower in the Nash equilibrium than in both Stackelberg equilibria. (b) Steady-state money growth and primary fiscal deficits are lower in the Stackelberg equilibrium with monetary leadership than in the Nash equilibrium. (c) Steady-state money growth and primary fiscal deficits are higher with fiscal leadership than in the Nash equilibrium.

Proof:

Expressions for steady-state debt, money growth and primary fiscal deficit are found in table 2. Result (a): the inequality $d^N(\infty)<d^M(\infty)$ reduces to $\lambda\tau(\delta-r)[(\delta-r)(\bar{f}-\bar{m}+r\bar{d}^M)+\lambda(\bar{d}^F-\bar{d}^M)]>0$ while $d^N(\infty)<d^M(\infty)$ reduces to $\lambda\tau(\delta-r)[(\delta-r)(\bar{f}-\bar{m}+r\bar{d}^F)+\tau(\bar{d}^M-\bar{d}^F)]>0$. Provided that the expressions inside the brackets are positive and given our earlier assumptions that $\lambda>0$, $\tau>0$ and $\delta>r$, part (a) results. Result (b): $m^M(\infty)<m^N(\infty)$ reduces to $\lambda(\lambda-r(\delta-r))[(\delta-r)(\bar{f}-\bar{m}+r\bar{d}^M)+\lambda(\bar{d}^F-\bar{d}^M)]>0$ while

[12] The system of linear differential equations, $\dot{\boldsymbol{x}}=\boldsymbol{Ax}(\boldsymbol{t})+\boldsymbol{b}$, is globally stable if all eigenvalues of the transition matrix $\boldsymbol{A}$ are negative. The system is saddlepoint stable if the number of backward-looking variables equals the number of negative eigenvalues and the number of the forward-looking variables equals the number of unstable eigenvalues. In all other cases the system is dynamically unstable and the transversality condition (2) is violated.

$f^M(\infty)<f^N(\infty)$ implies $\lambda\tau[(\delta-r)(\bar{f}-\bar{m}+r\bar{d}^M)+\lambda(\bar{d}^F-\bar{d}^M)]>0$. Note that $\lambda-r(\delta-r)>0$ is implied by the stability condition in case of Stackelberg leadership of the monetary authority. Result (c): $f^F(\infty)>f^N(\infty)$ reduces to $\tau(\tau-r(\delta-r))[(\delta-r)(\bar{f}-\bar{m}+r\bar{d}^F)+\tau(\bar{d}^M-\bar{d}^F)]>0$ while $m^F(\infty)>m^N(\infty)$ holds if $\lambda\tau[(\delta-r)(\bar{f}-\bar{m}+r\bar{d}^F)+\tau(\bar{d}^M-\bar{d}^F)]>0$. $\tau-r(\delta-r)>0$ is implied by the stability condition in case of Stackelberg leadership of the fiscal authority. □

The intuition behind proposition 1 is the following: the Stackelberg leader uses its strategic position to shift most of the adjustment burden to the Stackelberg follower. By moving first, it can constrain the action of the Stackelberg follower to a range that is optimal for himself. From the point of view of government debt stabilization the Stackelberg leadership of one of the authorities affects the performance negatively. Its Stackelberg leadership, however, allows the Stackelberg leader to perform better on its other objectives as compared to the Nash equilibrium. The higher adjustment burden from stabilizing debt that is put on the Stackelberg follower implies that it performs less on its other objectives. Part (b) and (c) indicate these effects.

3. Debt, deficits and money creation with a common currency

The introduction of a common currency issued and controlled by the ECB has important implications for public finance in the EU countries because the government budget constraints will link fiscal policies of the EU countries with the monetary policies of the ECB. In particular, the ECB controls money creation in the EU and redistributes the revenues from money creation towards the EU countries. Both the level of money creation and the redistribution of its revenues over the countries will have an impact on dynamics of fiscal deficits and government debt in the EMU. We explore the interaction between fiscal and monetary policies in the context of a monetary union of two countries. Country 1 and 2 receive a share of ECB base money creation or seigniorage, $m_E(t)$, according to their shares in the ECB denoted by θ and $1-\theta$, respectively. The dynamic government budget constraints of country 1 and 2 relate primary fiscal deficits in country 1 and 2, $f_1(t)$ and $f_2(t)$, monetization by the ECB, $m_E(t)$, interest payments on government debt, $rd_1(t)$ and $rd_2(t)$ and public debt accumulation, $\dot{d}_1$ and $\dot{d}_2$:

$$\dot{d}_1 = rd_1(t) + f_1(t) - \frac{\theta}{\omega} m_E(t) \tag{12a}$$

$$\dot{d}_2 = rd_2(t) + f_2(t) - \frac{1-\theta}{1-\omega} m_E(t) \tag{12b}$$

$d_1(t)$ and $f_1(t)$ are respectively country 1's outstanding government debt and primary fiscal deficit relative to country 1's GDP. Government debt and primary fiscal deficit of country 2 as a fraction of its GDP are denoted by $d_2(t)$ and $f_2(t)$. We assume that the high degree of integration of financial and goods markets in the EU makes that r is equal in both countries, and independent of the stock of outstanding government debt.

$m_E(t)$ denotes base money creation or seigniorage of the ECB, in relation to EU GDP. Dividing $\theta m_E(t)$ by the share, ω, of country 1 in EU GDP gives seigniorage revenues of country 1 in relation to GDP of country 1. If the distribution of seigniorage is based on the economic size of a country, i.e. if $\theta=\omega$ and $1-\theta=1-\omega$, seigniorage revenues that both countries receive will equal $m_E(t)$. Sibert (1994) endogenizes the seigniorage distribution $\{\theta,1-\theta\}$ in her modelling of the ECB. In the current analysis we assume that the seigniorage distribution is exogenously given[13].

We approach the issue of fiscal consolidation in the EMU in a 3-player differential game between the ECB and the two fiscal authorities in country 1 and 2. Consider again the following intertemporal loss function of the fiscal authority of country 1, which depends on the time profiles of the primary fiscal deficit and government debt:

$$L^{F_1}(t_0) = \frac{1}{2}\int_{t_0}^{\infty} \{(f_1(t) - \bar{f}_1)^2 + \lambda(d_1(t) - \bar{d}_1^F)^2\} e^{-\delta(t-t_0)} dt \tag{13}$$

The control variable of the fiscal authorities of country 1 is the primary fiscal deficit $f_1(t)$. The minimization problem of the fiscal authorities is carried out subject to the dynamic government budget constraint, (12a), the transversality condition on government debt (2) and the initial stock of government debt, $d_1(0)$.

The fiscal authorities in country 2 minimize a similar intertemporal loss function:

[13] One might consider the seigniorage distribution parameters as being the outcome of a Nash bargaining game between the countries who have a vote in the decision making process inside the ECB.

$$L^{F_2}(t) = \frac{1}{2}\int_0^\infty \{(f_2(t)-\bar{f}_2)^2 + \vartheta(d_2(t)-\bar{d}_2^F)^2\} e^{-\delta(t-t_0)} dt \tag{14}$$

subject to the dynamic government budget constraint of country 2 (12b), the transversality condition in (2) and the initial stock of government debt, $d_2(0)$[14].

The ECB selects money growth such as to minimize the following loss function:

$$L^E(t_0) = \frac{1}{2}\int_0^\infty \{(m_E(t)-\bar{m}_E)^2 + \tau[\omega(d_1(t)-\bar{d}_E)^2 + (1-\omega)(d_2(t)-\bar{d}_E)^2]\} e^{-\delta(t-t_0)} dt \tag{15}$$

subject to the dynamic government budget constraints in (12), the transversality conditions in (2) and given the initial stocks of debt $d_1(0)$ and $d_2(0)$. Money growth, $m_E(t)$, is the policy instrument of the ECB. Apart from a money growth -viz. inflation- target, the ECB is assumed to prefer low government debt in both countries. Government debt of country 1 and country 2 are weighted by the shares in EU GDP, ω and $1-\omega$, respectively. τ measures the degree of ECB conservativeness. If τ is equal to 0, the ECB only cares about price stability and it will not engage in monetization of debt. Such an extreme conservative ECB is not likely to arise in practice: countries are likely to use their voting power in the ECB and to form coalitions that seek to increase monetization of government debt or to change the seigniorage redistribution by the ECB if government debt is high[15].

We assume that the ECB cares about debt in each country separately, as tax distortions in each country are assumed to rise with the square of the tax rate. Therefore, monetary authorities prefer debt to be symmetrically distributed across countries. In that case monetary policy of the ECB is relatively sensitive to individual debt positions[16].

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[14] For analytical tractability in the remainder of the analysis we have assumed that fiscal players in both countries have the same rate of time preference.

[15] Cukierman (1992), lets the preferences and characteristics of the median EU country be decisive in ECB monetary policies.

[16] In van Aarle, Bovenberg and Raith (1995a) the ECB cares about average debt in the EU. In that case $[\omega(d_1(t)-\bar{d}_E)+(1-\omega)(d_2(t)-\bar{d}_E)]^2$ enters (15). Monetary policy in that case is less sensitive to individual debt positions. Results in the remainder of the analysis are not all independent of the form of the loss function of the ECB.

The same non-cooperative equilibria as studied in section 2 can be calculated in case where a monetary union with a common central bank has replaced national monetary policy. Appendix B provides the dynamic systems associated with the three equilibria in case of a monetary union with a common central bank. The dynamic systems that result are too complex to allow for an insightful analytical solution. In the case that we assume some symmetry between both fiscal players an attractive analytical solution, however, can be derived. In this section we therefore apply the method introduced by Aoki (1981) to derive an analytical solution to the dynamic systems.

This method decomposes system dynamics into an "average" and a "difference" part[17]. The average system describes the EU economy as a whole and its characteristics are similar to a situation with national monetary policy as analysed in section 2. The difference system describes cross-country differences in our 2-country EU. In our analysis, we consider country 1 to represent the Southern part of the EU that is assumed to have a higher initial stock of government debt, a higher primary fiscal deficit target, a higher monetary target (with national monetary policy) and higher debt targets than country 2 that represents the Northern part of the EU, i.e. $d_1(0) \geq d_2(0)$, $\bar{f}_1 \geq \bar{f}_2$, $\bar{m}_1 \geq \bar{m}_2$, $\bar{d}_1^M \geq \bar{d}_2^M$ and $\bar{d}_1^F \geq \bar{d}_2^F$. In debates on EMU concerns about this kind of asymmetries are often raised[18].

Studying the dynamics of the difference system is very useful if we are interested in the issue of convergence in debt and deficits in the EMU and in the absence of EMU. Details on actual convergence in the EU is found in the "Monetary and Economic Convergence Report" that the EC Commission (1995) has produced to inform about the progress that has been achieved in meeting the convergence criteria of the Maastricht Treaty.

The symmetry conditions that we need to impose to be able to decompose dynamics into an average and a difference part are that the two countries are of equal size and receive a seigniorage shares according to their economic size, i.e that $\omega=\theta=1-\omega=1-\theta=½$, and that both fiscal players attach equal priority to government debt stabilization, i.e. that $\lambda=\vartheta$. Assuming such symmetries, we can solve the dynamics of the

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[17] See Fukada (1993) for a *n*-country model where dynamics are decomposed into an average and *n*-1 difference systems.

[18] See e.g. arguments made in Alesina and Grilli (1993), Giovannini and Spaventa (1991) and Buiter and Kletzer (1991).

average and difference systems straightforwardly. We define averages and differences of a variable x as $x_A=½(x_1+x_2)$ and $x_D=x_1-x_2$, respectively. From the average and difference systems the country variables follow directly: $x_1=x_A+½x_D$ and $x_2=x_A-½x_D$.

Note that table 2 in section 2 also provides averages and differences of steady-state debt, primary fiscal deficits and money growth of a 2-country EU where national monetary policy still prevails. Averages in that case are found when replacing $\{\bar{f},\bar{m},\bar{d}^F,\bar{d}^M\}$ by $\{\bar{f}_A,\bar{m}_A,\bar{d}^F_A,\bar{d}^M_A\}$ in table 2 while differences result when replacing $\{\bar{f},\bar{m},\bar{d}^F,\bar{d}^M\}$ by $\{\bar{f}_D,\bar{m}_D,\bar{d}^F_D,\bar{d}^M_D\}$.

Appendix C gives the dynamic average and difference systems that result in the three different equilibria of the 2-country EMU. To ensure that the transversality condition (2) is satified and a process of explosive government debt is ruled out, we assume again that the dynamics of the average and relative systems are saddlepoint stable. With the aid of decomposition into averages and differences of variables it is straightforward to derive the steady-state of the Nash open-loop equilibrium and the Stackelberg open-loop equilibria with leadership of the ECB and of the fiscal players[19], respectively. Table 3 shows steady-state average and difference government debt, money growth and primary fiscal deficit in the monetary union:

[19] The current setup with a decomposition into averages and differences does not allow the fiscal authorities to differ in their strategic position versus the ECB. Thus, a situation where only one fiscal authority is Stackelberg leader or where the ECB is Stackelberg leader vis-à-vis only one fiscal authority and plays Nash with the other fiscal authority cannot be analyzed here. In this framework there is no direct conflict between both fiscal authorities and the equilibrium where both fiscal authorities have a strategic advantage vis-à-vis the ECB does not imply a form of unsustainable “Stackelberg warfare” between both. In van Aarle, Bovenberg and Raith (1995a) we allow for the possibility that the fiscal authorities form a fiscal coalition that plays non-cooperatively against the ECB.

Table 3 Steady state 2-country monetary union

	Nash open-loop	ECB leadership	Fiscal leadership
$d_A(\infty)$	$\frac{(\delta-r)^2(\bar{f}_A-\bar{m}_E)+(\delta-r)\lambda\bar{d}_A^F+(\delta-r)\tau\bar{d}_E}{\Delta_N}$	$\frac{(\delta-r)(r(\delta-r)-\lambda)(\bar{f}_A-\bar{m}_E)+(r(\delta-r)-\lambda)\lambda\bar{d}_A^F+r(\delta-r)\tau\bar{d}_E}{\Delta_M}$	$\frac{(\delta-r)(r(\delta-r)-\frac{\tau}{2})(\bar{f}_A-\bar{m}_E)+r(\delta-r)\lambda\bar{d}_A^F+(r(\delta-r)-\frac{\tau}{2})\tau\bar{d}_E}{\Delta_G}$
$d_D(\infty)$	$\frac{(\delta-r)[(\delta-r)\bar{f}_D+\lambda\bar{d}_D^F]}{\Delta_N}$	$\frac{(r(\delta-r)-\lambda)[(\delta-r)\bar{f}_D+\lambda\bar{d}_D^F]}{\Delta_M}$	$\frac{(\delta-r)[(r(\delta-r)-\frac{\tau}{2})\bar{f}_D+\lambda r\bar{d}_D^F]}{\Delta_G}$
$f_A(\infty)$	$\bar{f}_A-\frac{\lambda[(\delta-r)(\bar{f}_A-\bar{m}_E)+(r(\delta-r)-\tau)\bar{d}_A^F+\tau\bar{d}_E]}{\Delta_N}$	$\bar{f}_A-\frac{\lambda[(r(\delta-r)-\lambda)(\bar{f}_A-\bar{m}_E)+r(r(\delta-r)-\lambda-\tau)\bar{d}_A^F+\tau r\bar{d}_E]}{\Delta_M}$	$\bar{f}_A-\frac{\lambda r[(\delta-r)(\bar{f}_A-\bar{m}_E)+(r(\delta-r)-\tau)\bar{d}_A^F+\tau\bar{d}_E]}{\Delta_G}$
$f_D(\infty)$	$\bar{f}_D-\frac{\lambda[(\delta-r)\bar{f}_D+(r(\delta-r)-\tau)\bar{d}_D^F]}{\Delta_N}$	$\bar{f}_D-\frac{\lambda[(r(\delta-r)-\lambda)\bar{f}_D+r(r(\delta-r)-\lambda-\tau)\bar{d}_D^F]}{\Delta_M}$	$\bar{f}_D-\frac{\lambda r[(\delta-r)\bar{f}_D+(r(\delta-r)-\tau)\bar{d}_D^F]}{\Delta_G}$
$m_E(\infty)$	$\bar{m}_E+\frac{\tau[(\delta-r)(\bar{f}_A-\bar{m}_E)+\lambda\bar{d}_A^F+(r(\delta-r)-\lambda)\bar{d}_E]}{\Delta_N}$	$\bar{m}_E+\frac{\tau r[(\delta-r)(\bar{f}_A-\bar{m}_E)+\lambda\bar{d}_A^F+(r(\delta-r)-\lambda)\bar{d}_E]}{\Delta_M}$	$\bar{m}_E+\frac{\tau[(r(\delta-r)-\frac{\tau}{2})(\bar{f}_A-\bar{m}_E)+\lambda r\bar{d}_A^F+r(r(\delta-r)-\lambda-\frac{\tau}{2})\bar{d}_E]}{\Delta_G}$
Δ	$\Delta_N=[\lambda+\tau-r(\delta-r)](\delta-r)$	$\Delta_M=[2\lambda+\tau-r(\delta-r)]r(\delta-r)-\lambda^2$ $=r\Delta_N-\lambda(\lambda-r(\delta-r))$	$\Delta_G=[\lambda+\frac{3}{2}\tau-r(\delta-r)]r(\delta-r)-\frac{\tau^2}{2}$ $=r\Delta_N-\frac{\tau}{2}(\tau-r(\delta-r))$

The decomposition into averages and differences provided by table 3 is very useful as it gives immediate insight how asymmetries between countries affect the actual policies and outcomes in a monetary union. It is possible to formulate the following proposition on government debt, money growth and primary fiscal deficits in a monetary union:

Proposition 2

(a) Average steady-state debt is lower in the Nash equilibrium than in both Stackelberg equilibria. Money growth and average primary fiscal deficits are lower than in the Nash equilibrium if the ECB is Stackelberg leader and higher if the fiscal authorities are Stackelberg leader. (b) Differences in steady-state government debt are higher and differences in steady-state primary fiscal deficits are lower with Stackelberg leadership of the ECB than in the Nash-equilibrium. Differences in steady-state government debt and differences in steady-state primary fiscal deficit are higher with Stackelberg leadership of the fiscal authorities than in the Nash equilibrium, if $(\delta-r)\overline{f}_D>(\tau-r(\delta-r))\overline{d}^F_D$.

Proof:

(a) is found when reducing the same inequalities as in proposition 1, replacing the variables found in table 2 (steady-state with national monetary policy) by the variables found in table 3 (steady-state averages in a monetary union). Steady-state differences in government debt and primary fiscal deficits are found in table 3. The inequality $d^M_D(\infty)>d^N_D(\infty)$ reduces to: $\lambda\tau(\delta-r)[(\delta-r)\overline{f}_D+\lambda\overline{d}^F_D]>0$. Given our assumptions that $\lambda>0$, $\tau>0$, $\delta>r$, $\overline{f}_D>0$ and $\overline{d}^F_D>0$, this inequality holds throughout. $f^M_D(\infty)<f^N_D(\infty)$ can be rewritten as: $\lambda\tau[(\delta-r)\overline{f}_D+\lambda\overline{d}^F_D]>0$. $d^F_D(\infty)>d^N_D(\infty)$ implies that $\lambda\tau/2(\delta-r)[(\delta-r)\overline{f}_D+(r(\delta-r)-\tau)\overline{d}^F_D]>0$. $f^F_D(\infty)<f^N_D(\infty)$, finally, holds if $\tau/2(\tau-r(\delta-r))[(\delta-r)\overline{f}_D+(r(\delta-r)-\tau)\overline{d}^F_D]>0$. □

Part (a) of proposition 2 offers a generalization of proposition 1 to a monetary union and suggests that similar incentives prevail in a monetary union as with national monetary policy. According to (b) an independent ECB increases the discrepancies in steady-state government debt in the monetary union as compared to the Nash equilibrium although steady-state differences in primary fiscal deficit are reduced. These effects occur because an independent ECB leaves most of the adjustment burden with the fiscal authorities in the participating countries. Country 1 faces a higher adjustment burden than country 2 and a non-accommodating ECB implies an increase in steady-state differences in debt and a reduction in steady-state differences in primary fiscal deficits. A monetary union with a dependent ECB is likely to increase further the discrepancies in government indebtedness and primary fiscal deficits, as compared to the Nash equilibrium.

The initial stock of government debt of influences monetary and fiscal policies during the adjustment towards steady-state but does not affect the steady-state itself. The country with a higher than average initial stock of government debt has a larger adjustment burden than the country with a below average initial stock of debt. Consequently, its primary fiscal deficit during the adjustment towards steady-state is lower than the average fiscal deficit whereas the other country can afford to have an above average fiscal deficit. Moreover, with national monetary policies the first country needs to generate more seigniorage revenues than average whereas the other country will have below average money growth. In the long-run, however, the impact of the initial stock of government debt of both countries vanishes and debt, deficits and money growth depend exclusively on the preference parameters, the net interest rate of debt and the rate of time preferences as tables 2 and 3 reveal. The numerical analysis of section 5 illustrates the adjustment towards steady-state for a numerical example.

4. A comparison with pre-EMU monetary and fiscal policy interaction

It is interesting to compare the performance of monetary and fiscal policy under the EMU regime, as analysed in section 3 with a "pre-EMU" regime with national central banks as analysed in section 2. In our setup a full monetary union implies that monetary policy has been centralized at a European level. In the model of section 3 a monetary union results in both countries having the same rate of money growth -set by the ECB-, i.e. $m_1(t)=m_2(t)=m_E(t)$, and the same rate of inflation. A regime with national monetary policies on the other hand could represent a “two-speed” EMU in which groups of countries in the EMU follow their own preferred monetary policy, independent of the other EMU-part, and where a full monetary union is not yet achieved. Such a comparison allows us to analyse the consequences of a monetary union between countries that differ in initial government indebtedness and/or policy targets.

The model of fiscal and monetary policy interaction in a monetary union with 2 countries of section 3, reduces to the model of monetary and fiscal policy interaction with a national monetary authority of section 2, if θ=ω=1 (in case of country 1) or 1−θ= 1−ω=1 (in case of country 2). Consider first the case where the countries do not differ in debt and primary fiscal deficit targets, i.e. $\bar{d}_1^F=\bar{d}_2^F=\bar{d}_A^F$ and $\bar{f}_1=\bar{f}_2=\bar{f}_A$, and have the same initial stock of government debt. Assume that the ECB and the former national monetary authorities have the same money growth and debt targets and the same degree of conservativeness, i.e. $\bar{m}_1=\bar{m}_2=\bar{m}_A=\bar{m}_E$, $\bar{d}_1^M=\bar{d}_2^M=\bar{d}_A^M=\bar{d}_E$ and τ equal with national monetary policy and an ECB. In that case the difference part of the dynamic system vanishes and the average system will describe dynamics of debt, money growth and primary fiscal deficit in both countries. We can formulate the following proposition:

Proposition 3

Compared with national monetary policies, a monetary union between identical countries will not change dynamics of government debt, primary fiscal deficit and money growth in the Nash open-loop equilibrium and in the Stackelberg open-loop with the monetary authority leading. A monetary union results in higher steady-state debt, money growth and primary fiscal deficits if the fiscal authorities are Stackelberg leader.

Proof:

In case of symmetric countries, $\bar{f}_A=\bar{f}$, $\bar{d}_A=\bar{d}^F$ and if we assume that $\bar{m}_E=\bar{m}$, $\bar{d}_E=\bar{d}^M$ (ECB has same targets as national central bank), steady-state debt, money growth (inflation) and primary fiscal deficit coincide in case of national monetary policy and a monetary union if the Nash open-loop equilibrium or the Stackelberg equilibrium with monetary leadership prevails (cf. table 2 and 3). Steady-state debt is higher in the monetary union with fiscal leadership than in the 1-country case with fiscal leadership, $d_A^F(\infty)>d^F(\infty)$, if $\tau\lambda r(\delta-r)[(\delta-r)(\bar{f}-\bar{m}+r\bar{d}^F)+\tau(\bar{d}^M-\bar{d}^F)]>0$. Steady-state money growth is higher in a monetary union with fiscal leadership than in the 1-country case with fiscal leadership, $m_E^F(\infty)>m^F(\infty)$ if $\tau\lambda r[(\delta-r)(\bar{f}-\bar{m}+r\bar{d}^F)+\tau(\bar{d}^M-\bar{d}^F)]>0$. Steady-state primary fiscal deficits and money growth are higher in a monetary union with fiscal leadership than in the 1-country case with fiscal leadership, $f_A^F(\infty)>f^F(\infty)$, if $\tau(\tau-r(\delta-r))[(\delta-r)(\bar{f}-\bar{m}+r\bar{d}^F)+\tau(\bar{d}^M-\bar{d}^F)]>0$. Provided that the term in brackets is positive, proposition 3 holds. □

With identical countries the difference system vanishes. The average system -which is identical to the closed-economy dynamic system (provided that the preference function of the ECB equals the preference function of the national central bank) in the Nash equilibrium and the Stackelberg equilibrium with monetary leadership- then describes dynamics of government debt, money growth and primary fiscal deficits in both countries. In case of Stackelberg leadership of the fiscal authorities, the ECB is exploited by two fiscal authorities instead of one as with a national central bank. A monetary union with a dependent ECB will therefore produce even higher debt, inflation and fiscal deficits than with national central banks that are dependent.

Consider next a monetary union of asymmetric countries: assume that the primary fiscal deficit target, the money growth target of the monetary authority before the monetary union and the debt target of country 1 exceed the targets of country 2, i.e. $\bar{f}_D>0$, $\bar{m}_D>0$ and $\bar{d}_D>0$. The decomposition into averages and differences enables us to calculate the differences between a EU with national monetary policy and a monetary union between the same countries. We derive the following result, assuming that the money and debt target of the ECB coincides with the average targets of the national central banks, i.e. $\bar{m}_E=\bar{m}_A$ and $\bar{d}_E=\bar{d}^M_A$:

Proposition 4

Compared with national monetary policies, a monetary union of asymmetric countries: (a) will not change average steady-state debt, money growth and primary fiscal deficits in the Nash open-loop equilibrium and with Stackelberg leadership of the ECB. With Stackelberg leadership of the fiscal authorities a monetary union increases average debt, money creation and primary fiscal deficits, (b) will increase steady-state differences in government debt and decrease steady-state differences in primary fiscal deficit in the Nash equilibrium if $(\delta-r)\bar{m}_D>\tau\bar{d}^M_D$ *and in the case of Stackelberg leadership of the monetary authority. In case of Stackelberg leadership of the fiscal authorities the impact of a monetary union on steady-state differences in government debt and primary fiscal deficits is ambiguous.*[20]

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[20] If the fiscal authorities are Stackelberg leader, a monetary union will increase steady-state differences in government debt if

Proof:

Average debt, money growth and primary fiscal deficits with national monetary policy are found when replacing $\bar{f}$ with $\bar{f}_A$, $\bar{m}$ with $\bar{m}_A$, $\bar{d}^F$ with $\bar{d}^F_A$ and $\bar{d}^M$ with $\bar{d}^M_A$ in table 2. If we assume that $\bar{m}_E=\bar{m}_A$ and $\bar{d}_E=\bar{d}^M_A$ the expressions of steady-state average debt, money growth and primary fiscal deficit in the monetary union and with national monetary policy coincide in case of the Nash equilibrium and the Stackelberg equilibrium with monetary leadership. With fiscal leadership, steady-state average debt is higher in a monetary union than with national monetary policies if $\tau\lambda r(\delta-r)\ [(\delta-r)(\bar{f}_A-\bar{m}_A+r\bar{d}^F_A)+\tau(\bar{d}^M-\bar{d}^F)]>0$. Steady-state money growth is higher in a monetary union if $\tau\lambda r[(\delta-r)(\bar{f}_A-\bar{m}_A+r\bar{d}^F_A)+\tau(\bar{d}^M-\bar{d}^F)]>0$ and primary fiscal deficits are higher if $\tau(\tau-r(\delta-r))[(\delta-r)(\bar{f}_A-\bar{m}_A+r\bar{d}^F_A)+\tau(\bar{d}^M-\bar{d}^F)]>0$. Given our earlier assumptions, (a) results. Steady-state differences with national monetary policy are found when replacing $\{\bar{f},\bar{m},\bar{d}^F,\bar{d}^M\}$ by $\{\bar{f}_D,\bar{m}_D,\bar{d}^F_D,\bar{d}^M_D\}$ in table 2. Writing out the inequalities one finds that the difference in steady-state debt is higher in a monetary union than with national monetary policies if $(\delta-r)[(\delta-r)\bar{m}_D-\tau\bar{d}^M_D]/\Delta_N>0$ in the Nash equilibrium and if $(\delta-r)[(r(\delta-r)-\lambda)\bar{m}_D-\tau r\bar{d}^M_D]/\Delta_M>0$ in case of monetary leadership. The steady-state difference in primary fiscal deficit is smaller with a monetary union than with national monetary policy if $-\lambda[(\delta-r)\bar{m}_D-\tau\bar{d}^M_D]/\Delta_N<0$ in the Nash equilibrium and if $-\lambda[(r(\delta-r)-\lambda)\bar{m}_D-\tau r\bar{d}^M_D]/\Delta_M<0$ in case of monetary leadership. □

Part (a) of proposition 4 confirms much of the fear implicit in the Report of the Delors Committee of the inflationary consequences of a dependant ECB -i.e. Stackelberg leadership of the fiscal authorities- that is forced to monetize a substantial part of the financing requirements of undisciplined national fiscal authorities. According to (b), whether a monetary union increases steady-state convergence in debt and deficits depends on the mode of interaction between ECB and fiscal authorities and on the particular set of model parameters that is deemed realistic.

$-\frac{\tau}{2}(\tau-r(\delta-r))(\delta-r)[(r(\delta-r)-\frac{\tau}{2})\bar{f}_D+r\lambda\bar{d}^F_D]+\Delta_G[\frac{\tau}{2}(\delta-r)\bar{f}_D+(r(\delta-r)-\tau)((\delta-r)\bar{m}_D-\tau\bar{d}^M_D)]>0$ and steady-state differences in primary fiscal deficit if

$-\frac{\tau}{2}(\tau-r(\delta-r))[(\delta-r)\bar{f}_D+(r(\delta-r)-\tau)\bar{d}^F_D]+\Delta_G[(\delta-r)\bar{m}_D-\tau\bar{d}^M_D]>0$

The information on the properties of the average and difference systems in proposition 4 enables us to calculate the impact of entering a monetary union from an individual country perspective as country variables can easily be calculated from the average and difference systems. The average system is not affected by entering the monetary union in case of the Nash equilibrium and with Stackelberg leadership of the monetary authority and the impact of entering a monetary union is easy to determine in that case. In case of country 1 (2), steady-state debt is higher (lower) and steady-state money growth and primary fiscal deficits are lower (higher) in a monetary union than with national monetary policy in case of the Nash equilibrium and Stackelberg leadership of the monetary authority[21]. Rows 2 and 3 of table 4 give the individual country effects from entering a monetary union with asymmetric countries in the Nash case and with an independent ECB, respectively.

The individual country effects in case of Stackelberg leadership of the fiscal authorities are ambiguous in principle. The more symmetric both countries become the less important are the effects of a monetary union on steady-state differences and the average effect dominates. If (twice) the difference effect is smaller than the average effect, a monetary union will increase steady-state government debt money creation and primary fiscal deficits in both countries. The first line (I) in row 4 of table 4 gives the individual country effects if the average effect dominates.

If (twice) the differential effect of a monetary union dominates also other outcomes may result in the long run. In that case, the effects of a monetary union with a dependent ECB are opposite in both countries. If e.g. a monetary union with a dependent ECB increases steady-state differences in government debt and primary fiscal deficits[22], country 1 experiences an increase in debt and primary fiscal deficit and a decrease in money growth whereas country 2 will experience lower debt and deficits but higher money growth. The second line (II) in row 4 of table 4 gives the individual country effects if the difference effect dominates.

[21] We assume that the conditions necessary for proposition 4 to hold -as stated in its proof- are satisfied.

[22] See the conditions in footnote 17 for this to be the case.

Table 4 Individual country effects

	$d_1(\infty)$	$d_2(\infty)$	$f_1(\infty)$	$f_2(\infty)$	$m_1(\infty)$	$m_2(\infty)$
Nash	+	−	−	+	−	+
Mon. Stack	+	−	−	+	−	+
Fiscal (I) Stack (II)	+ +/−	+ +/−	+ +/-	+ +/−	+ −	+ +

Important for these results are the assumptions that money growth and debt targets of the ECB are the average of the former national central banks, i.e. $\overline{m}_E=\overline{m}_A$ and $\overline{d}_E=\overline{d}_A^M$ and that the ECB gives the same priority to debt stabilization than the former national banks, i.e. τ does not change with a monetary union. From the perspective of country 1, the ECB is monetizing less of the fiscal deficits than its former national central bank. The loss of seigniorage revenues when entering the monetary union puts additional pressure on government debt accumulation. The higher steady-state debt is met with lower steady-state primary fiscal deficits in the Nash equilibrium and the Stackelberg equilibrium with monetary leadership. Country 2 experiences the opposite: the ECB is more accommodating than its former national central bank. The increase in seigniorage revenues allows a reduction in steady-state debt and an increase in primary fiscal deficits in both equilibria.

While proposition 3 and 4 compare Stackelberg leadership of the fiscal authorities in a 2-country monetary union with the single country case it can be generalized to a *n*-country monetary union as well: in that case the term $\tau/2$ in the last column of table 2 is replaced by τ/n. We can formulate the following result:

Proposition 5

A monetary union with a weak ECB creates higher average debt, average primary fiscal deficits and money growth/inflation if more countries participate. There is, however, a limit to the number of countries that can join a monetary union with a weak ECB if dynamic instability is to be precluded. As long as the number of countries is less than: $n^*=\tau(\tau-r(\delta-r))/r\Delta_N$, *a monetary union with a dependent ECB is not dynamically unstable. The maximum number of countries that can participate increases if the ECB cares more about debt stabilization and the fiscal authorities less, i.e. if* τ *is high and* λ *is* *low. If the union with a weak ECB consists of more countries than n^*, the only feasible solution is in fact the Nash equilibrium.*

Proof:

Average steady-state government debt, primary fiscal deficits and money growth with a monetary union of n-countries is found when replacing $\tau/2$ by τ/n in the last column of table 3. An increase in n increases Δ_G (which is negative in case of stability) and by that average steady-state government debt, primary fiscal deficits and money growth in the monetary union. Taking the limit for $n \to \infty$, gives the Nash equilibrium. Dynamic stability of a n-country monetary union holds as long as $r\Delta_N - \tau/n(\tau - r(\delta - r)) < 0$. If $n \geq \tau(\tau - r(\delta - r))/r\Delta_N$, this inequality fails to hold. An increase of τ raises n^* as $\frac{\partial n^*}{\partial \tau} = \frac{2\tau - r(\delta - r)}{r\Delta_N} + \frac{\tau(\tau - r(\delta - r)}{r\Delta_N(\lambda + \tau - r(\delta - r))}$ is positive given our assumption that $\tau - r(\delta - r) > 0$. An increase in λ decreases n^* as $\frac{\partial n^*}{\partial \lambda} = -\frac{\tau(\tau - r(\delta - r)}{r\Delta_N(\lambda + \tau - r(\delta - r))}$ is negative. □

If the fiscal authorities care little about debt stabilization, i.e. if λ is small, they do not use their strategic advantage to force high monetization by a weak ECB to reduce government debt. On the other hand, if the ECB cares much about debt stabilization, i.e. if τ is large, less of the adjustment burden is postponed to the future. In both cases the number of countries that can participate in the monetary union without generating an unstable process of government debt accumulation increases. Figure 1 pictures n^* as a function of λ and τ (δ and r have been put equal to 0.1 and 0.05[23]), respectively:

[23] A higher rate of time preference reduces n^* if $r(\delta - r)(2\tau - r(\delta - r)) - \tau(\tau + \lambda) < 0$. A higher rate of (net) interest on government debt reduces n^* if $(\tau - (\delta - r)^2)r(\lambda + \tau - r(\delta - r)) - (\tau - r(\delta - r))[(2r - \delta)r + \lambda + \tau - r(\delta - r)] < 0$.

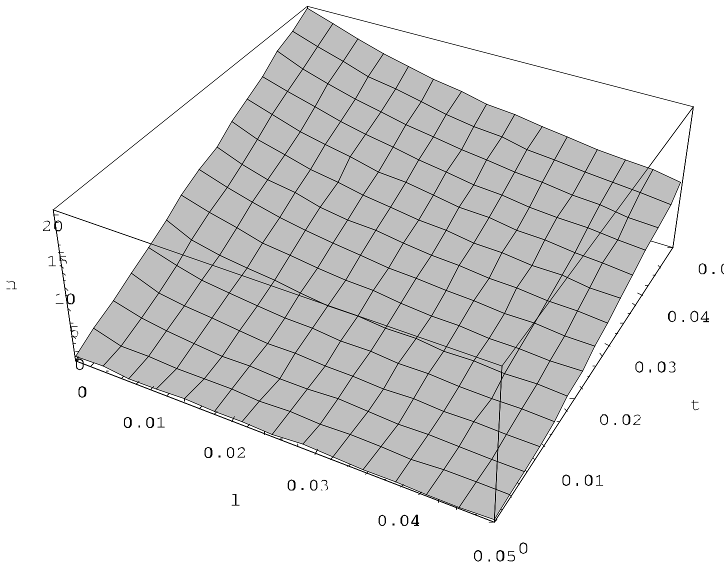


figure 1 n^* as a function of λ and τ.

Several scenarios are possible for a country that decides to enter the monetary union[24]: a country with a dependent national bank, i.e. Stackelberg leadership of the fiscal authority could enter a monetary union with an independent ECB, an ECB that plays Nash with the fiscal authorities or a monetary union with also a dependent ECB. Alternatively, a country with an independent central bank could enter a monetary union with a dependent ECB. In all these cases, the consequences from giving up monetary independence and entering a monetary union can be calculated with the framework of sections 2 and 3. Changes in the institutional setting in which monetary and fiscal authorities interact may have a profound impact on the country's performance on debt and inflation stabilization. In particular, the change from one extreme to the other extreme is likely to cause abrupt changes.

[24] See Currie (1992) and Levine and Pearlman (1992) for such scenario' approaches. Beetsma and Bovenberg (1995) address the question under which circumstances a monetary union is feasible in the sense of improving welfare (or at least not deteriorating) of its participants.

5. A numerical example

To illustrate the theoretical results we obtained in sections 2-4 consider next a stylized numerical example of a two-country EU. Table 5 gives the values of the model parameters that we use throughout our example:

Table 5 A numerical example

	country 1	country 2	average/ ECB	difference
$d(0)$	1.0	0.6	0.8	0.4
$\bar{m}$	0.015	0.000	0.005	0.01
$\bar{f}$	0.015	0.005	0.01	0.01
$\bar{d}^F$	0.60	0.60	0.60	0
$\bar{d}^M$	0.60	0.60	0.60	0
λ	0.03	τ	0.02	
r	0.04	δ	0.1	

Initial debt, money growth - and primary fiscal deficit targets are chosen conform the asymmetries between country 1 and 2 we introduced in section 3: country 1 has a higher stock of initial debt and a higher money growth and deficit target than country 2. The debt target was set for all policymakers to 0.6, the target value of the Maastricht Treaty. We impose again the symmetry conditions regarding $\{\lambda,\tau,\delta,r\}$. Assume that this setting describes a 2-country EU with country 1 representing the high debt countries and country 2 the low debt countries. Table 6 calculates steady-state debt, primary fiscal deficits and money growth both in case where the countries would maintain national monetary policies (section 2) and in case where a monetary union is formed between both countries (section 3).

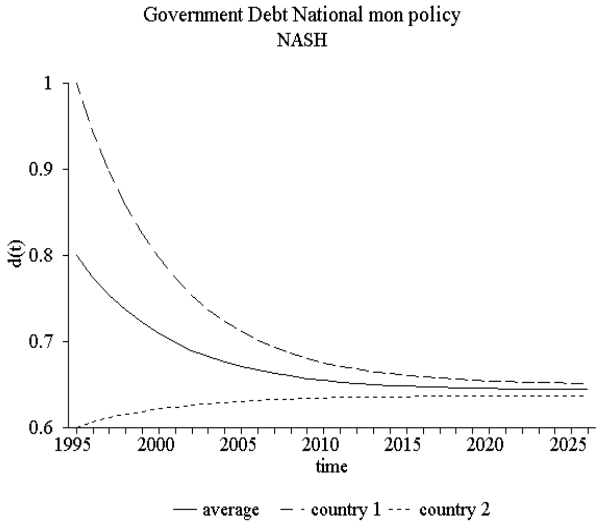
Table 6 National monetary policy vs. EMU

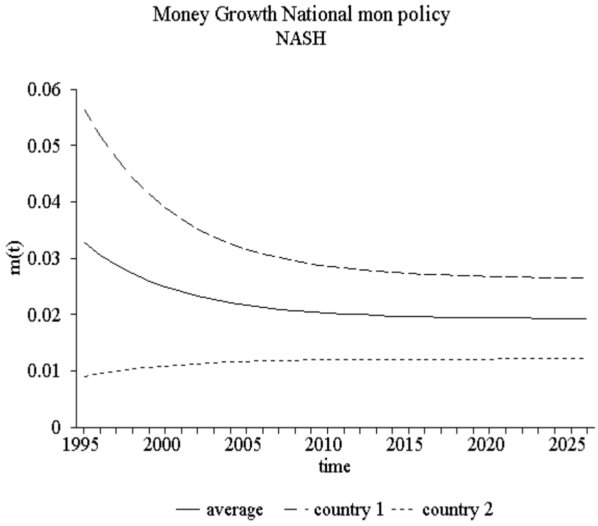
(% GDP)	Nash		Mon Stack leader		Fisc Stack leader	
	Nat. CB	ECB/EMU	Nat. CB	ECB/EMU	Nat. CB	ECB/EMU
$d_A(\infty)$	0.643	0.643	0.679	0.679	0.751	0.851
$d_D(\infty)$	0.013	0.025	0.023	0.046	0.044	0.148
$d_1(\infty)$	0.649	0.655	0.690	0.702	0.773	0.925
$d_2(\infty)$	0.649	0.630	0.667	0.656	0.729	0.777
$f_A(\infty)$	-0.006	-0.006	-0.024	-0.024	0.025	0.055
$f_D(\infty)$	0.014	0.007	0.008	-0.003	0.023	0.043
$f_1(\infty)$	0.000	-0.003	-0.020	-0.023	0.037	0.076
$f_2(\infty)$	-0.013	-0.010	-0.029	-0.026	0.014	0.033
$m_A(\infty)$	0.019	0.019	0.003	0.003	0.055	0.089
$m_D(\infty)$	0.014	0	0.007	0	0.025	0
$m_1(\infty)$	0.026	0.019	0.007	0.003	0.068	0.089
$m_2(\infty)$	0.012	0.019	-0.002	0.003	0.043	0.089

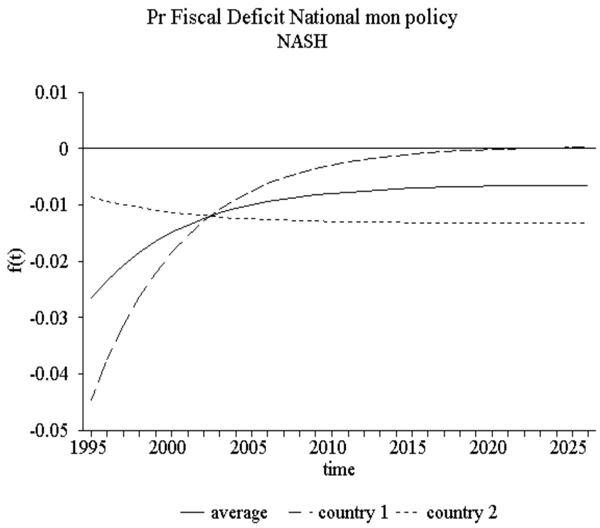
A careful look at table 6 confirms the analytical results found in propositions 1-4 and the results in table 4. The magnitude of the differences in steady-state government debt, primary fiscal deficits and money growth between the different equilibria and between the case of national monetary policies and an EMU with an ECB are relatively small, reflecting a strong degree of convergence in debt, deficits and money growth. This result depends on the assumptions in this numerical example that all policymakers have the same debt target and that the differences in primary fiscal deficit and money growth targets are relatively small. Of course increasing the differences in debt, deficit and money growth targets will produce stronger divergences in the long-run.

Figure 1 shows dynamics of government debt, primary fiscal deficit and money growth with national monetary policy.

(1a) Nash

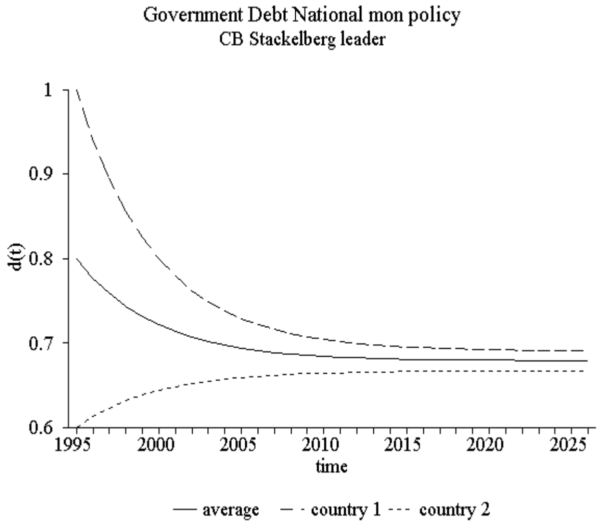


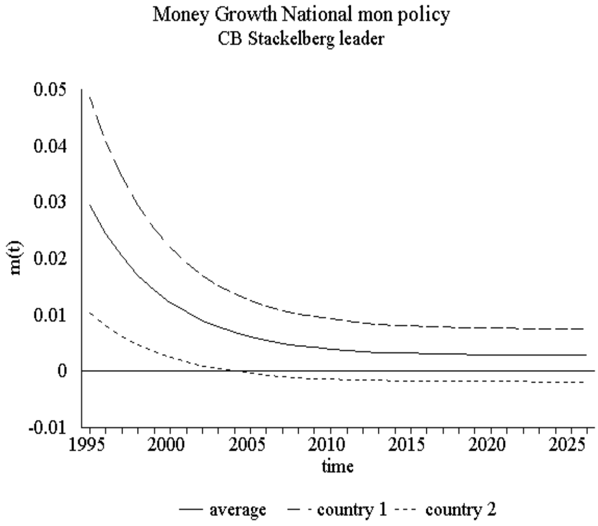


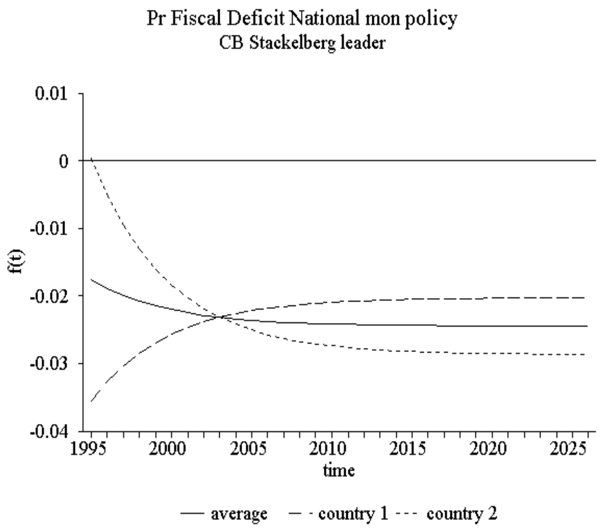


Initial debt in country 1 is below the steady-state level, by that inducing a small initial decline in money growth (compared to steady-state money growth) and a small increase in primary fiscal deficits. Since country 2 its initial debt is above steady-state an initial increase in money growth and a decline in primary fiscal deficits is evoked.

(1b) Independant national central banks

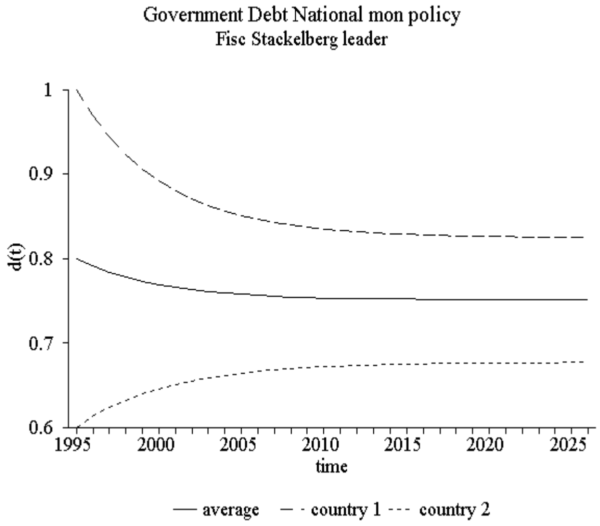


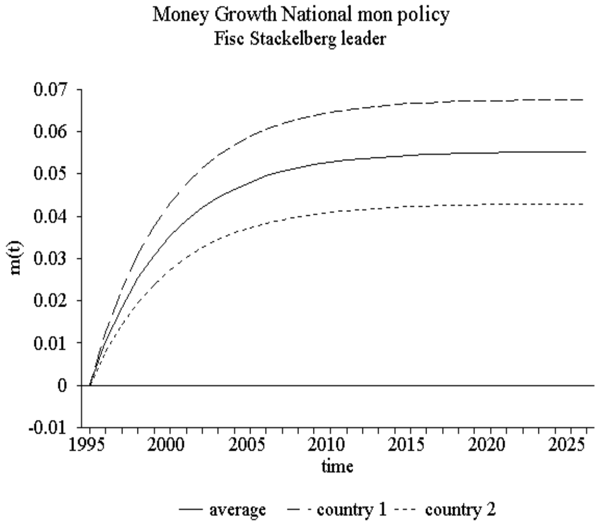


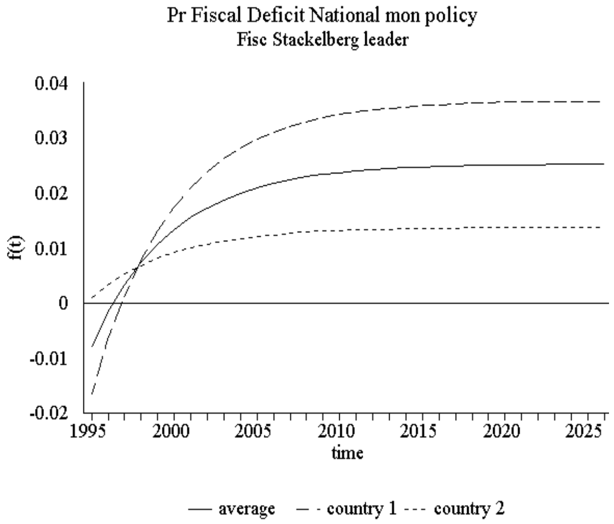


Compared to the Nash equilibrium (1a), an independant monetary authority succeeds in shifting much of the adjustment burden to the fiscal authorities. The lower money growth that results, however, is obtained at the cost of higher debt and lower fiscal deficits, a result proved earlier in proposition 1.

(c) Dependant national central banks



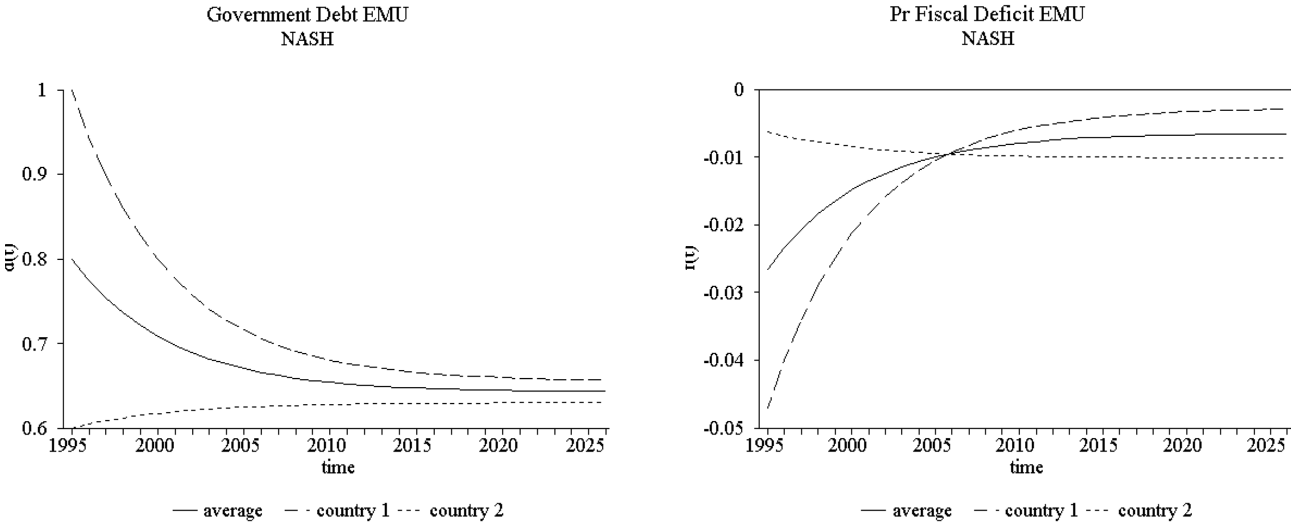




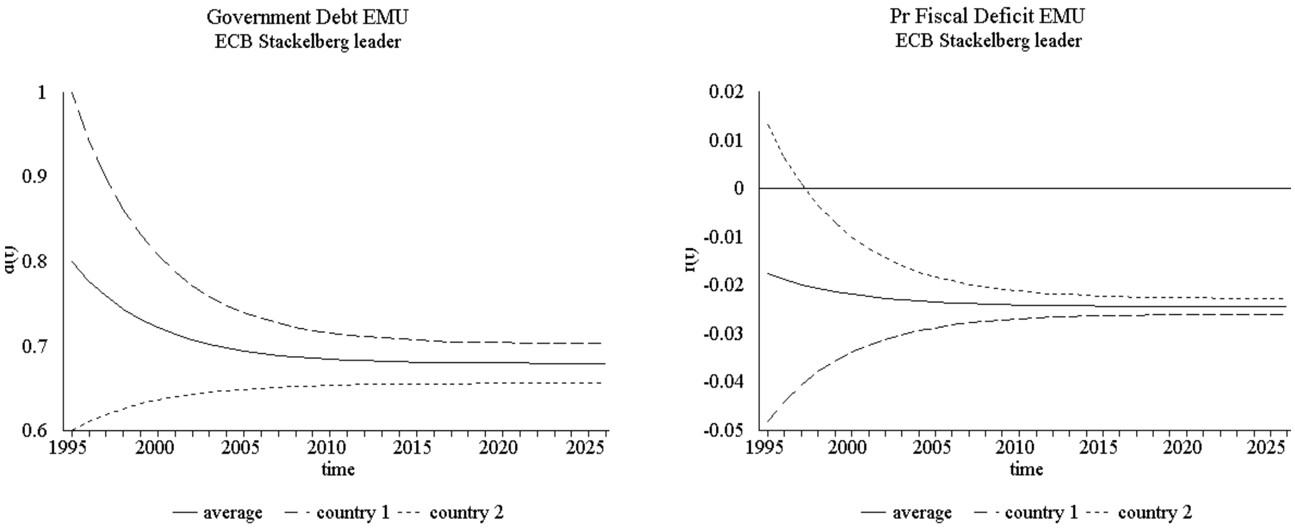
A dependant central bank produces less debt stabilization, higher money growth and deficits, compared to the Nash equilibrium as the fiscal authority shifts as much as possible the adjustment burden from debt stabilization to the monetary authority.

Figure 2 shows dynamics of government debt, primary fiscal deficit and money growth with a monetary union:

(2a) Nash

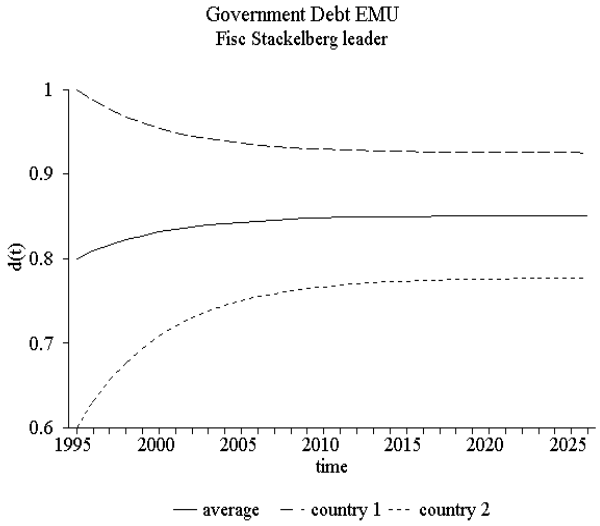


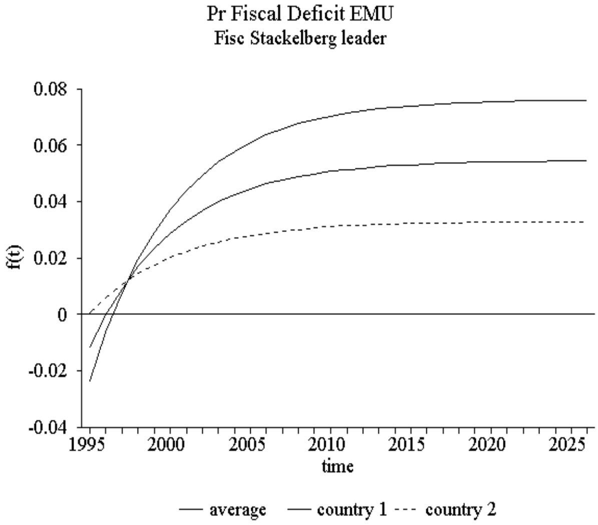
(2b) Independant ECB



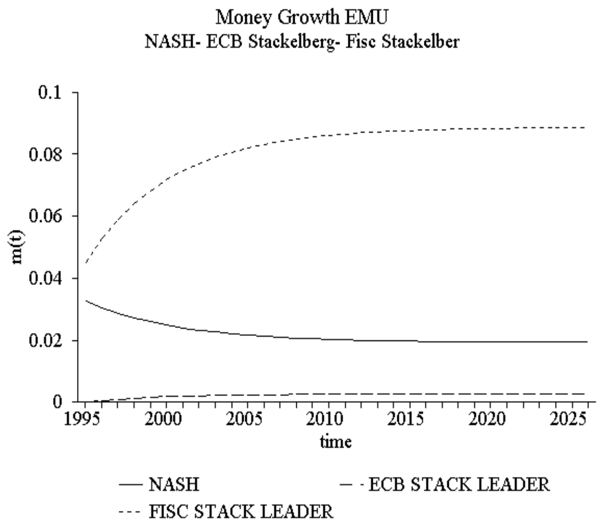
Comparing figs. (2a) and (2b) and money growth as found in (2d) with national monetary policy (1a) and (2a) gives the result (a) of proposition 4 that the monetary union between both countries does not affect average debt, deficit and money growth in case of the Nash and Stackelberg equilibrium with the monetary authority leading.

(2c) Dependant ECB





(2d) Money Growth in a Monetary Union



Comparing (2c) and money growth with a weak ECB in (2d) with (1c) shows the increase in (average) debt, deficits and money growth when moving from national monetary policy to a monetary union.

Conclusions

The creation of a monetary union with a common central bank in the EU has aroused much attention. This paper focused on the consequences of a monetary union from the perspective of government debt stabilization. The problem of government debt stabilization has played a central role in the debate on monetary and fiscal convergence initiated by the Report of the Delors Committee.

Following Tabellini (1986), the problem of government debt stabilization was analysed as a differential game between a monetary authority who controls monetization and a fiscal player that controls primary fiscal deficits. The problem of government debt stabilization was first analysed in the context of national monetary policies. Next, we considered a monetary union in which monetary policy has been centralized. The consequences of establishing a monetary union between two countries with asymmetric policy preferences were investigated. The analysis revealed the importance of the degree of independence of national central banks and the ECB on the dynamics of government debt, money growth and primary fiscal deficits. A dependent ECB was shown to risk a strong inflationary-, and deficit bias in the EU.

Future research effort could be directed at modifying some of the assumptions made in the analysis. In particular one may want to change the assumption that the preferences of the ECB are an average of the individual countries, that the (net) interest rate on government debt is equal in both countries and independent of the level of debt and that the rate of time preference of all players is equal. Also the role of the seigniorage distribution function and the different scenarios that a country faces when entering a monetary union could be subject of a more detailed analysis.

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Appendix A System dynamics with national monetary policy

I The Nash open-loop equilibrium

The dynamic system of the Nash open-loop equilibrium is given by:

$$\begin{bmatrix} \dot{d} \\ \dot{\mu}^F \\ \dot{\mu}^M \end{bmatrix} = \begin{bmatrix} r & -1 & -1 \\ -\lambda & \delta-r & 0 \\ -\tau & 0 & \delta-r \end{bmatrix} \begin{bmatrix} d(t) \\ \mu^F(t) \\ \mu^M(t) \end{bmatrix} + \begin{bmatrix} \bar{f}-\bar{m} \\ \lambda\bar{d}^F \\ \tau\bar{d}^M \end{bmatrix} \tag{A.1}$$

The dynamic system consists of one backward-looking variables, $d(t)$, and two forward-looking variables, $\mu^F(t)$ and $\mu^M(t)$. Saddlepoint stability requires that $\det(A)=(\delta-r)(r(\delta-r)-\lambda-\tau)<0$. The inverse of the transient matrix A is equal to:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} (\delta-r)^2 & \delta-r & \delta-r \\ \lambda(\delta-r) & r(\delta-r)-\kappa & \lambda \\ \kappa(\delta-r) & \kappa & r(\delta-r)-\lambda \end{bmatrix} \tag{A.2}$$

The steady-state of the Nash open-loop equilibrium can be written as:

$$\begin{aligned} d(\infty) &= \frac{(\delta-r)^2(\bar{f}-\bar{m})+\lambda(\delta-r)\bar{d}^F+\tau(\delta-r)\bar{d}^M}{\Delta_N} \\ \mu^F(\infty) &= \frac{\lambda(\delta-r)(\bar{f}-\bar{m})+(r(\delta-r)-\tau)\lambda\bar{d}^F+\lambda\tau\bar{d}^M}{\Delta_N} \\ \mu^M(\infty) &= \frac{\tau(\delta-r)(\bar{f}-\bar{m})+\tau\lambda\bar{d}^F+(r(\delta-r)-\lambda)\tau\bar{d}^M}{\Delta_N} \end{aligned} \tag{A.3}$$

in which Δ_N equals $-\det(A)=(\delta-r)(\lambda+\tau-r(\delta-r))$. Using the first order conditions in (6) and (8), we can write $f(\infty)$ and $m(\infty)$ as:

$$\begin{aligned} f(\infty) &= \bar{f} - \mu^F(\infty) = \bar{f} - \frac{\lambda(\delta-r)(\bar{f}-\bar{m})+(r(\delta-r)-\tau)\lambda\bar{d}^F+\lambda\tau\bar{d}^M}{\Delta_N} \\ m(\infty) &= \bar{m} + \mu^M(\infty) = \bar{m} + \frac{\tau(\delta-r)(\bar{f}-\bar{m})+\lambda\tau\bar{d}^F+(r(\delta-r)-\lambda)\tau\bar{d}^M}{\Delta_N} \end{aligned} \tag{A.4}$$

II The Stackelberg open-loop equilibrium with the monetary authority as leader

The dynamic system of the Stackelberg open-loop equilibrium with the monetary authority leading can be written as:

$$\begin{bmatrix} d \\ \dot{\mu}^F \\ \dot{\mu}^M \\ \dot{\rho}^M \end{bmatrix} = \begin{bmatrix} r & -1 & -1 & 0 \\ -\lambda & \delta-r & 0 & 0 \\ -\tau & 0 & \delta-r & \lambda \\ 0 & 0 & 1 & r \end{bmatrix} \begin{bmatrix} d(t) \\ \mu^F(t) \\ \mu^M(t) \\ \rho^M(t) \end{bmatrix} + \begin{bmatrix} \bar{f}-\bar{m} \\ \lambda\bar{d}^F \\ \tau\bar{d}^M \\ 0 \end{bmatrix} \tag{A.5}$$

The dynamic system consists of two backward-looking variables, $d(t)$ and $\rho^M(t)$, and two forward-looking variables, $\mu^F(t)$ and $\mu^M(t)$. Saddlepoint stability requires that $\det(A)=\lambda^2-(2\lambda+\kappa-r(\delta-r))r(\delta-r)<0$. The inverse of A equals:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} (\delta-r)(r(\delta-r)-\lambda) & r(\delta-r)-\lambda & r(\delta-r) & -\lambda(\delta-r) \\ \lambda(r(\delta-r)-\lambda) & r(r(\delta-r)-\lambda-\tau) & \lambda r & -\lambda^2 \\ \tau r(\delta-r) & \tau r & r(r(\delta-r)-\lambda) & \lambda(\lambda-r(\delta-r)) \\ -\tau(\delta-r) & -\tau & -(r(\delta-r)-\lambda) & (\delta-r)(r(\delta-r)-\lambda-\tau) \end{bmatrix} \tag{A.6}$$

The steady-state of the Stackelberg open-loop equilibrium with monetary leadership can be written as:

$$\begin{aligned} d(\infty) &= \frac{(\delta-r)(r(\delta-r)-\lambda)(\bar{f}-\bar{m})+(r(\delta-r)-\lambda)\lambda\bar{d}^F+r(\delta-r)\tau\bar{d}^M}{\Delta_M} \\ \mu^F(\infty) &= \frac{\lambda(r(\delta-r)-\lambda)(\bar{f}-\bar{m})+r(r(\delta-r)-\lambda-\tau)\lambda\bar{d}^F+\lambda r\tau\bar{d}^M}{\Delta_M} \\ \mu^M(\infty) &= \frac{\tau r(\delta-r)(\bar{f}-\bar{m})+\tau r\lambda\bar{d}^F+r(r(\delta-r)-\lambda)\tau\bar{d}^M}{\Delta_M} \end{aligned} \tag{A.7}$$

in which Δ_M equals $-\det(A)=(2\lambda+\tau-r(\delta-r))r(\delta-r)-\lambda^2$. Using the first order conditions in (6) and (8), we can write $f(\infty)$ and $m(\infty)$ as:

$$\begin{aligned} f(\infty) &= \bar{f}-\mu^F(\infty) = \bar{f}-\frac{\lambda(r(\delta-r)-r)(\bar{f}-\bar{m})+r(r(\delta-r)-\lambda-\tau)\lambda\bar{d}^F+\lambda r\tau\bar{d}^M}{\Delta_M} \\ m(\infty) &= \bar{m}+\mu^M(\infty) = \bar{m}+\frac{\tau r(\delta-r)(\bar{f}-\bar{m})+\tau r\lambda\bar{d}^F+r(r(\delta-r)-\lambda)\tau\bar{d}^M}{\Delta_M} \end{aligned} \tag{A.8}$$

III The Stackelberg open-loop equilibrium with the fiscal authority as leader

The dynamic system of the Stackelberg open-loop equilibrium with the fiscal authorities leading equals:

$$\begin{bmatrix} \dot{d} \\ \dot{\mu}^F \\ \dot{\mu}^M \\ \dot{\rho}^F \end{bmatrix} = \begin{bmatrix} r & -1 & -1 & 0 \\ -\lambda & \delta-r & 0 & \tau \\ -\tau & 0 & \delta-r & 0 \\ 0 & 1 & 0 & r \end{bmatrix} \begin{bmatrix} d(t) \\ \mu^F(t) \\ \mu^M(t) \\ \rho^F(t) \end{bmatrix} + \begin{bmatrix} \bar{f}-\bar{m} \\ \lambda\bar{d}^F \\ \tau\bar{d}^M \\ 0 \end{bmatrix} \tag{A.9}$$

The dynamic system consists of two backward-looking variables, $d(t)$ and $\rho^F(t)$, and two forward-looking variables, $\mu^F(t)$ and $\mu^M(t)$. Saddlepoint stability requires that $\det(A)=\tau^2-[\lambda+2\tau-r(\delta-r)]r(\delta-r)$ where A is the transient matrix of (B.8). The inverse of the transient matrix A is equal to:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} (\delta-r)(r(\delta-r)-\tau) & r(\delta-r) & r(\delta-r)-\tau & -\tau(\delta-r) \\ \lambda r(\delta-r) & r(r(\delta-r)-\tau) & r\lambda & -\tau(r(\delta-r)-\tau) \\ \tau(r(\delta-r)-\tau) & \tau r & r(r(\delta-r)-\lambda-\tau) & -\tau^2 \\ -\lambda(\delta-r) & \tau-r(\delta-r) & -\lambda & (\delta-r)(r(\delta-r)-\lambda-\tau) \end{bmatrix} \tag{A.10}$$

The steady-state of the Stackelberg open-loop equilibrium with the fiscal authority acting as Stackelberg leader can be written as:

$$
\begin{aligned}
d(\infty) &= \frac{(\delta-r)(r(\delta-r)-\tau)(\bar{f}-\bar{m})+r(\delta-r)\lambda\bar{d}^{F}+(r(\delta-r)-\tau)\tau\bar{d}^{M}}{\Delta_F}\\
\mu^{F}(\infty) &= \frac{\lambda r(\delta-r)(\bar{f}-\bar{m})+r(r(\delta-r)-\tau)\lambda\bar{d}^{F}+r\lambda\tau\bar{d}^{M}}{\Delta_F}\\
\mu^{M}(\infty) &= \frac{\tau(r(\delta-r)-\tau)(\bar{f}-\bar{m})+r\tau\lambda\bar{d}^{F}+r(r(\delta-r)-\lambda-\tau)\tau\bar{d}^{M}}{\Delta_F}
\end{aligned}
\tag{A.11}
$$

in which Δ_F equals $-\det(A)=[\lambda+2\tau-r(\delta-r)]r(\delta\text{-}r)\text{-}\tau^2$. Using the first order conditions in (6) and (8), we can write $f(\infty)$ and $m(\infty)$ as:

$$
\begin{aligned}
f(\infty) &= \bar{f}-\mu^{F}(\infty) = \bar{f}-\frac{\lambda r(\delta-r)(\bar{f}-\bar{m})+r(r(\delta-r)-\tau)\lambda\bar{d}^{F}+r\lambda\tau\bar{d}^{M}}{\Delta_F}\\
m(\infty) &= \bar{m}+\mu^{M}(\infty) = \bar{m}+\frac{\tau(r(\delta-r)-\tau)(\bar{f}-\bar{m})+r\tau\lambda\bar{d}^{F}+r(r(\delta-r)-\lambda-\tau)\tau\bar{d}^{M}}{\Delta_F}
\end{aligned}
\tag{A.12}
$$

Appendix B Dynamic systems of the Nash open-loop and Stackelberg open-loop equilibria with a monetary union

The Nash open-loop equilibrium is given by:

$$
\begin{bmatrix} \dot{d}_1 \\ \dot{d}_2 \\ \dot{\mu}_1^{F1} \\ \dot{\mu}_2^{F2} \\ \dot{\mu}_1^{E} \\ \dot{\mu}_2^{E} \end{bmatrix}
=
\begin{bmatrix}
r & 0 & -1 & 0 & -\frac{\theta^2}{\omega^2} & -\frac{\theta(1-\theta)}{\omega(1-\omega)} \\
0 & r & 0 & -1 & -\frac{\theta(1-\theta)}{\omega(1-\omega)} & -\frac{(1-\theta)^2}{(1-\omega)^2} \\
-\lambda & 0 & \delta-r & 0 & 0 & 0 \\
0 & -\vartheta & 0 & \delta-r & 0 & 0 \\
-\tau\omega & 0 & 0 & 0 & \delta-r & 0 \\
0 & -\tau(1-\omega) & 0 & 0 & 0 & \delta-r
\end{bmatrix}
\begin{bmatrix} d_1(t) \\ d_2(t) \\ \mu_1^{F1}(t) \\ \mu_2^{F2}(t) \\ \mu_1^{E} \\ \mu_2^{E} \end{bmatrix}
+
\begin{bmatrix} \bar{f}_1-\frac{\theta}{\omega}\bar{m}_E \\ \bar{f}_2-\frac{1-\theta}{1-\omega}m_E \\ \lambda\bar{d}_1 \\ \vartheta\bar{d}_2 \\ \tau\omega\bar{d}_E \\ \tau(1-\omega)\bar{d}_E \end{bmatrix}
\tag{B.1}
$$

The Stackelberg open-loop equilibrium with the ECB acting as a Stackelberg leader is described by:

$$
\begin{bmatrix} \dot{d}_1 \\ \dot{d}_2 \\ \dot{\mu}_1^{F1} \\ \dot{\mu}_2^{F2} \\ \dot{\mu}_1^{E} \\ \dot{\mu}_2^{E} \\ \dot{\rho}_1^{E} \\ \dot{\rho}_2^{E} \end{bmatrix}
=
\begin{bmatrix}
r & 0 & -1 & 0 & -\frac{\theta^2}{\omega^2} & -\frac{\theta(1-\theta)}{\omega(1-\omega)} & 0 & 0 \\
0 & r & 0 & -1 & -\frac{\theta(1-\theta)}{\omega(1-\omega)} & -\frac{(1-\theta)^2}{(1-\omega)^2} & 0 & 0 \\
-\lambda & 0 & \delta-r & 0 & 0 & 0 & 0 & 0 \\
0 & -\vartheta & 0 & \delta-r & 0 & 0 & 0 & 0 \\
-\tau\omega & 0 & 0 & 0 & \delta-r & 0 & \lambda & 0 \\
0 & -\tau(1-\omega) & 0 & 0 & 0 & \delta-r & 0 & \vartheta \\
0 & 0 & 0 & 0 & 1 & 0 & r & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & r
\end{bmatrix}
\begin{bmatrix} d_1(t) \\ d_2(t) \\ \mu_1^{F1}(t) \\ \mu_2^{F2}(t) \\ \mu_1^{E}(t) \\ \mu_2^{E}(t) \\ \rho_1^{E}(t) \\ \rho_2^{E}(t) \end{bmatrix}
+
\begin{bmatrix} \bar{f}_1-\frac{\theta}{\omega}\bar{m}_E \\ \bar{f}_2-\frac{1-\theta}{1-\omega}m_E \\ \lambda\bar{d}_1 \\ \vartheta\bar{d}_2 \\ \tau\omega\bar{d}_E \\ \tau(1-\omega)\bar{d}_E \\ 0 \\ 0 \end{bmatrix}
\tag{B.2}
$$

The Stackelberg equilibrium with the fiscal players acting as a Stackelberg leader towards the ECB, is given by:

$$\begin{bmatrix} \dot{d}_1 \\ \dot{d}_2 \\ \dot{\mu}_1^{F1} \\ \dot{\mu}_2^{F2} \\ \dot{\mu}_1^{E} \\ \dot{\mu}_2^{E} \\ \dot{\rho}_1^{F1} \\ \dot{\rho}_2^{F2} \end{bmatrix} = \begin{bmatrix} r & 0 & -1 & 0 & -\frac{\theta^2}{\omega^2} & -\frac{\theta(1-\theta)}{\omega(1-\omega)} & 0 & 0 \\ 0 & r & 0 & -1 & -\frac{\theta(1-\theta)}{\omega(1-\omega)} & -\frac{(1-\theta)^2}{(1-\omega)^2} & 0 & 0 \\ -\lambda & 0 & \delta-r & 0 & 0 & 0 & \tau\omega & 0 \\ 0 & -\vartheta & 0 & \delta-r & 0 & 0 & \tau(1-\omega) & 0 \\ -\tau\omega & 0 & 0 & 0 & \delta-r & 0 & 0 & 0 \\ 0 & -\tau(1-\omega) & 0 & 0 & 0 & \delta-r & 0 & 0 \\ 0 & 0 & \frac{\theta^2}{\omega^2} & 0 & 0 & 0 & r & 0 \\ 0 & 0 & 0 & \frac{(1-\theta)^2}{(1-\omega)^2} & 0 & 0 & 0 & r \end{bmatrix} \begin{bmatrix} d_1(t) \\ d_2(t) \\ \mu_1^{F1}(t) \\ \mu_2^{F2}(t) \\ \mu_1^{E}(t) \\ \mu_2^{E}(t) \\ \rho_1^{F1}(t) \\ \rho_2^{F2}(t) \end{bmatrix} + \begin{bmatrix} \bar{f}_1 - \frac{\theta}{\omega}\bar{m}_E \\ \bar{f}_2 - \frac{1-\theta}{1-\omega} m_E \\ \lambda \bar{d}_1 \\ \vartheta \bar{d}_2 \\ \tau\omega \bar{d}_E \\ \tau(1-\omega)\bar{d}_E \\ 0 \\ 0 \end{bmatrix} \quad \text{(B.3)}$$

Appendix C Average and difference systems of the different equilibria with a monetary union

Defining averages of a variable x as $x_A=½(x_1+x_2)$ and differences in x as $x_D=x_1-x_2$, we can decompose the dynamics of (B1), (B2) and (B3) into an average and a difference if we impose the symmetry conditions $\theta=\omega=1-\theta=1-\omega=½$ and $\lambda=\vartheta$.

The average system of the Nash open-loop equilibrium equals:

$$\begin{bmatrix} \dot{d}_A \\ \dot{\mu}_A^F \\ \dot{\mu}_A^E \end{bmatrix} = \begin{bmatrix} r & -1 & -2 \\ -\lambda & \delta-r & 0 \\ -\frac{\tau}{2} & 0 & \delta-r \end{bmatrix} \begin{bmatrix} d_A(t) \\ \mu_A^F(t) \\ \mu_A^E(t) \end{bmatrix} + \begin{bmatrix} \bar{f}_A - \bar{m}_E \\ \lambda \bar{d}_A^F \\ \frac{\tau}{2}\bar{d}_E \end{bmatrix} \quad \text{(C.1)}$$

whereas its difference part is defined as:

$$\begin{bmatrix} \dot{d}_D \\ \dot{\mu}_D^F \\ \dot{\mu}_D^E \end{bmatrix} = \begin{bmatrix} r & -1 & -2 \\ -\lambda & \delta-r & 0 \\ -\frac{\tau}{2} & 0 & \delta-r \end{bmatrix} \begin{bmatrix} d_D(t) \\ \mu_D^F(t) \\ \mu_D^E(t) \end{bmatrix} + \begin{bmatrix} \bar{f}_D \\ \lambda \bar{d}_D^F \\ 0 \end{bmatrix} \quad \text{(C.2)}$$

The inverse of the transition matrix A that governs system dynamics of the average and difference systems equals:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} (\delta-r)^2 & \delta-r & \delta-r \\ \lambda(\delta-r) & r(\delta-r)-\tau & 2\lambda \\ \frac{\tau}{2}(\delta-r) & \frac{\tau}{2} & r(\delta-r)-\lambda \end{bmatrix} \quad \text{(C.3)}$$

where $\det(A)$ equals $(\delta-r)(r(\delta-r)-\lambda-\tau)$. The steady-state of the Nash open-loop equilibrium, therefore, can be written as:

$$
\begin{aligned}
d_A(\infty) &= \frac{(\delta-r)\,[(\delta-r)(\bar{f}_A-\bar{m}_E)+\lambda d_A^F+\tau d_E]}{\Delta_N}\\
\mu_A^F(\infty) &= \frac{\lambda\,[(\delta-r)(\bar{f}_A-\bar{m}_E)+(r(\delta-r)-\tau)d_A^F+\tau d_E]}{\Delta_N}\\
\mu_A^E(\infty) &= \frac{\frac{\tau}{2}\,[(\delta-r)(\bar{f}_A-\bar{m}_E)+\lambda d_A^F+(r(\delta-r)-\lambda)d_E]}{\Delta_N}\\
d_D(\infty) &= \frac{(\delta-r)\,[(\delta-r)\bar{f}_D+\lambda d_D^F]}{\Delta_N}\\
\mu_D^F(\infty) &= \frac{\lambda[(\delta-r)\bar{f}_D+(r(\delta-r)-\tau)\,d_D^F]}{\Delta_N}
\end{aligned}
\tag{C.4}
$$

in which Δ_N equals $-\det(A)$. Using the first order conditions, we can write $f_A(\infty)$, $m_E(\infty)$ and $f_D(\infty)$ as:

$$
\begin{aligned}
f_A(\infty) &= \bar{f}_A-\mu_A^F(\infty)=\bar{f}_A-\frac{\lambda\,[(\delta-r)(\bar{f}_A-\bar{m}_E)+(r(\delta-r)-\tau)\bar{d}_A^F+\tau\bar{d}_E]}{\Delta_N}\\
m_E(\infty) &= \bar{m}_E+2\,\mu_A^E(\infty)=\bar{m}_E+\frac{\tau\,[(\delta-r)(\bar{f}_A-\bar{m}_E)+\lambda\bar{d}_A^F+(r(\delta-r)-\lambda)\bar{d}_E]}{\Delta_N}\\
f_D(\infty) &= \bar{f}_D-\mu_D^F(\infty)=\bar{f}_D-\frac{\lambda\,[(\delta-r)\bar{f}_D+(r(\delta-r)-\tau)\,d_D^F]}{\Delta_N}
\end{aligned}
\tag{C.5}
$$

The dynamic system consists of one backward-looking variable, $d_A(t)$, and two forward-looking variables, $\mu_A^F(t)$ and $\mu_A^E(t)$. If we impose the condition that $\Delta_N>0$ the system will be saddlepoint stable.

The average system of the Stackelberg open-loop equilibrium with the ECB leading equals:

$$
\begin{bmatrix}\dot{d}_A\\ \dot{\mu}_A^F\\ \dot{\mu}_A^E\\ \dot{\rho}_A^E\end{bmatrix}=
\begin{bmatrix} r & -1 & -2 & 0\\ -\lambda & \delta-r & 0 & 0\\ -\frac{\tau}{2} & 0 & \delta-r & \lambda\\ 0 & 0 & 1 & r\end{bmatrix}
\begin{bmatrix} d_A(t)\\ \mu_A^F(t)\\ \mu_A^E(t)\\ \rho_A^E(t)\end{bmatrix}+
\begin{bmatrix}\bar{f}_A-\bar{m}_E\\ \lambda\bar{d}_A^F\\ \frac{\tau}{2}\bar{d}_E\\ 0\end{bmatrix}
\tag{C.6}
$$

whereas its difference part is given by:

$$
\begin{bmatrix}\dot{d}_D\\ \dot{\mu}_D^F\\ \dot{\mu}_D^E\\ \dot{\rho}_D^E\end{bmatrix}=
\begin{bmatrix} r & -1 & -2 & 0\\ -\lambda & \delta-r & 0 & 0\\ -\frac{\tau}{2} & 0 & \delta-r & \lambda\\ 0 & 0 & 1 & r\end{bmatrix}
\begin{bmatrix} d_D(t)\\ \mu_D^F(t)\\ \mu_D^E(t)\\ \rho_D^E(t)\end{bmatrix}+
\begin{bmatrix}\bar{f}_D\\ \lambda\bar{d}_D^F\\ 0\\ 0\end{bmatrix}
\tag{C.7}
$$

The inverse of the transition matrix A that governs system dynamics of the average and difference systems equals:

$$
A^{-1}=\frac{1}{\det(A)}
\begin{bmatrix}
(\delta-r)(r(\delta-r)-\lambda) & r(\delta-r)-\lambda & 2r(\delta-r) & -2\lambda(\delta-r)\\
\lambda(r(\delta-r)-\lambda) & r(r(\delta-r)-\lambda-\tau) & 2\lambda r & -2\lambda^2\\
\frac{\tau}{2}r(\delta-r) & \frac{\tau}{2}r & r(r(\delta-r)-\lambda) & -\lambda(r(\delta-r)-\lambda)\\
-\frac{\tau}{2}(\delta-r) & -\frac{\tau}{2} & -(r(\delta-r)-\lambda) & (\delta-r)(r(\delta-r)-\lambda-\tau)
\end{bmatrix}
\tag{C.8}
$$

in which $\det(A)$ equals $\lambda^2-(2\lambda+\tau-r(\delta-r))r(\delta-r)$. Steady-state debt and the steady-state co-state variables associated with government debt can be written as:

$$
\begin{aligned}
d_A(\infty) &= \frac{(\delta-r)(r(\delta-r)-\lambda)(\bar{f}_A-\bar{m}_E)+(r(\delta-r)-\lambda)\lambda\bar{d}_A^F+r(\delta-r)\tau\bar{d}_E}{\Delta_M} \\
\mu_A^F(\infty) &= \frac{\lambda[(r(\delta-r)-\lambda)(\bar{f}_A-\bar{m}_E)+r(r(\delta-r)-\lambda-\tau)\bar{d}_A^F+\tau r\bar{d}_E]}{\Delta_M} \\
\mu_A^E(\infty) &= \frac{\frac{\tau}{2}r[(\delta-r)(\bar{f}_A-\bar{m}_E)+r\lambda d_A^F+(r(\delta-r)-\lambda)\bar{d}_E]}{\Delta_M} \\
d_D(\infty) &= \frac{(\delta-r)(r(\delta-r)-\lambda)\bar{f}_D+(r(\delta-r)-\lambda)\lambda\bar{d}_D^F]}{\Delta_M} \\
\mu_D^F(\infty) &= \frac{\lambda[(r(\delta-r)-\lambda)\bar{f}_D+(r(\delta-r)-\lambda-\tau)r\bar{d}_D^F]}{\Delta_M}
\end{aligned}
\tag{C.9}
$$

with $\Delta_M=-\det(A)$. Using the first order conditions, we can write $f_A(\infty)$, $m_E(\infty)$ and $f_D(\infty)$ as:

$$
\begin{aligned}
f_A(\infty) &= \bar{f}_A-\mu_A^F(\infty)=\bar{f}_A-\frac{\lambda[(r(\delta-r)-\lambda)(\bar{f}_A-\bar{m}_E)+(r(\delta-r)-\lambda-\tau))r\bar{d}_A^F+r\tau d_E]}{\Delta_M} \\
m(\infty) &= \bar{m}_E+2\,\mu_A^E(\infty)=\bar{m}_E+\frac{\tau r[(\delta-r)(\bar{f}_A-\bar{m}_E)+\lambda\bar{d}_A^F+(r(\delta-r)-\lambda)d_E]}{\Delta_M} \\
f_D(\infty) &= \bar{f}_D-\mu_D^F(\infty)=\bar{f}_D-\frac{\lambda[(r(\delta-r)-\lambda)\bar{f}_D+(r(\delta-r)-\lambda-\tau))r\bar{d}_D^F]}{\Delta_M}
\end{aligned}
\tag{C.10}
$$

The dynamic system consists of two backward-looking variables, $d_A(t)$ and $\rho_A^E(t)$, and two forward-looking variables, $\mu_A^F(t)$ and $\mu_A^E(t)$. If we impose the condition that $\Delta_M<0$ the system will be (saddlepoint) stable.

The average system of the Stackelberg open-loop equilibrium with the fiscal authorities leading equals:

$$
\begin{bmatrix} \dot{d}_A \\ \dot{\mu}_A^F \\ \dot{\mu}_A^E \\ \dot{\rho}_A^F \end{bmatrix} = \begin{bmatrix} r & -1 & -2 & 0 \\ -\lambda & \delta-r & 0 & \frac{\tau}{2} \\ -\frac{\tau}{2} & 0 & \delta-r & 0 \\ 0 & 1 & 0 & r \end{bmatrix} \begin{bmatrix} d_A(t) \\ \mu_A^F(t) \\ \mu_A^E(t) \\ \rho_A^F(t) \end{bmatrix} + \begin{bmatrix} \bar{f}_A-\bar{m}_E \\ \lambda\bar{d}_A^F \\ \frac{\tau}{2}\bar{d}_E \\ 0 \end{bmatrix}
\tag{C.11}
$$

whereas its difference part is given by:

$$
\begin{bmatrix} \dot{d}_D \\ \dot{\mu}_D^F \\ \dot{\mu}_D^E \\ \dot{\rho}_D^F \end{bmatrix} = \begin{bmatrix} r & -1 & -2 & 0 \\ -\lambda & \delta-r & 0 & \frac{\tau}{2} \\ -\frac{\tau}{2} & 0 & \delta-r & 0 \\ 0 & 1 & 0 & r \end{bmatrix} \begin{bmatrix} d_D(t) \\ \mu_D^F(t) \\ \mu_D^E(t) \\ \rho_D^F(t) \end{bmatrix} + \begin{bmatrix} \bar{f}_D \\ \lambda\bar{d}_D^F \\ 0 \\ 0 \end{bmatrix}
\tag{C.12}
$$

The inverse of the transition matrix A that governs system dynamics of the average and difference systems equals:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} (\delta - r)(r(\delta - r) - \frac{\tau}{2}) & r(\delta - r) & 2r(\delta - r) - \tau & -\frac{\tau}{2}(\delta - r) \\ \lambda r(\delta - r) & r(r(\delta - r) - \tau) & 2r\lambda & -\frac{\tau}{2}(r(\delta - r) - \tau) \\ \frac{\tau}{2}(r(\delta - r) - \frac{\tau}{2}) & \frac{\tau}{2}r & r(r(\delta - r) - \lambda - \frac{\tau}{2}) & -(\frac{\tau}{2})^2 \\ -\lambda(\delta - r) & \tau - r(\delta - r) & -2\lambda & (\delta - r)(r(\delta - r) - \lambda - \tau) \end{bmatrix} \tag{C.13}$$

in which $\det(A)$ equals $\tau^2/2-[\lambda+3/2\tau-r(\delta\text{-}r)]r(\delta-r)$. Steady-state debt and the steady-state co-state variables associated with government debt can be written as:

$$\begin{aligned} d_A(\infty) &= \frac{(\delta - r)(r(\delta - r) - \frac{\tau}{2})(\bar{f}_A - \bar{m}_E) + r(\delta - r)\lambda \bar{d}_A^F + (r(\delta - r) - \frac{\tau}{2})\tau \bar{d}_E}{\Delta_G} \\ \mu_A^F(\infty) &= \frac{\lambda r[(\delta - r)(\bar{f}_A - \bar{m}_E) + (r(\delta - r) - \tau)\bar{d}_A^F + \tau \bar{d}_E]}{\Delta_G} \\ \mu_A^E(\infty) &= \frac{\frac{\tau}{2}[(r(\delta - r) - \frac{\tau}{2})(\bar{f}_A - \bar{m}_E) + \lambda r \bar{d}_A^F + r(r(\delta - r) - \lambda - \frac{\tau}{2})\bar{d}_E]}{\Delta_G} \\ d_D(\infty) &= \frac{(\delta - r)[(r(\delta - r) - \frac{\tau}{2})\bar{f}_D + \lambda r \bar{d}_D^F]}{\Delta_G} \\ \mu_D^F(\infty) &= \frac{\lambda r[(\delta - r)\bar{f}_D + (r(\delta - r) - \tau)\bar{d}_D^F]}{\Delta_G} \end{aligned} \tag{C.14}$$

in which Δ_G=det(A). Using the first order conditions in (9), (11) and (13), we can write $f_A(\infty)$, $m_E(\infty)$ and $f_D(\infty)$ as:

$$\begin{aligned} f_A(\infty) &= \bar{f}_A - \mu_A^F(\infty) = \bar{f}_A - \frac{\lambda r[(\delta - r)(\bar{f}_A - \bar{m}_E) + (r(\delta - r) - \tau)\bar{d}_A^F + \tau d_E]}{\Delta_G} \\ m_E(\infty) &= \bar{m}_E + 2\mu_A^E(\infty) = \bar{m}_E + \frac{\tau[(r(\delta - r) - \frac{\tau}{2})(\bar{f}_A - \bar{m}_E) + \lambda r \bar{d}_A^F + r(r(\delta - r) - \lambda - \frac{\tau}{2})d_E]}{\Delta_G} \\ f_D(\infty) &= \bar{f}_D - \mu_D^F(\infty) = \bar{f}_D - \frac{\lambda r[(\delta - r)\bar{f}_D + (r(\delta - r) - \tau)\bar{d}_D^F]}{\Delta_G} \end{aligned} \tag{C.15}$$

The dynamic system consists of two backward-looking variables, $d_A(t)$ and $\rho_A^F(t)$, and two forward-looking variables, $\mu_A^F(t)$ and $\mu_A^E(t)$. If we impose the condition that Δ_F<0 the system will be (saddlepoint) stable.